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ELEMENTARY ALGEBRA

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JOHN WALLIS (1616–1703)

Professor at the University of Oxford. In 1685 he published a *Treatise of Algebra* which was the means of making the science of algebra more widely known in England.

ELEMENTARY ALGEBRA

Second Year Course

BY

FLORIAN CAJORI

COLORADO COLLEGE

AND

LETITIA R. ODELL

NORTH SIDE HIGH SCHOOL, DENVER

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PREFACE

This book contains a brief review of the fundamental operations of algebra followed by a thorough presentation of the topics usually included in the work of the third half year. The material is so arranged that the choice of topics for review or advance study may be easily made.

The book contains not only a large number of practical problems but also practical applications of graphs. Material for drill in the manipulation of exponents and radicals will be found in Chapter VII. Great pains have been taken to make the subject of logarithms accessible to beginners. The most difficult part in learning to compute with logarithms is the process of "interpolation." Of the various arrangements of logarithmic tables which have been suggested, we have selected the one which renders interpolation easiest. The chapter on logarithms is introduced earlier than usual, on account of the great practical importance of the subject. If desired, it can be taken up later, after Chapter VI. The concept of a *function* does not receive isolated and abstract treatment; it is presented as a fundamental idea in proportion, variation, and graphics. Its connection with problems of everyday life is firmly established.

The aims of the First Year Book have been kept in view in the preparation of this text. Emphasis is thrown upon clearness of exposition and the use of expressions which recall the axiomatic processes involved. Continued stress has been laid upon oral exercises.

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Several of the practical problems given in the text were suggested by the perusal of an English book, T. Percy Nunn's *Exercises in Algebra* (*including Trigonometry*), Part I, 1913.

The authors have received help from several teachers. Special mention should be made of Mr. E. A. Cummings of the North Side High School in Denver, who has worked most of the exercises and offered valuable suggestions.

FLORIAN CAJORI.
LETITIA R. ODELL.

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ELEMENTARY ALGEBRA

SECOND YEAR COURSE

CHAPTER I

ELEMENTARY DEFINITIONS AND OPERATIONS

FUNDAMENTAL DEFINITIONS

1. In arithmetic, numbers are commonly represented by Hindu-Arabic numerals. In algebra, numbers are represented also by letters.

Any combination of numerals, letters and symbols of operation, which stands for a number, is called an *algebraic expression.*

An algebraic expression may consist of parts which are separated by the + or − signs; these parts, with the signs immediately preceding them, are called *terms.*

Thus, in $+a^2 - 2ab + 3b^2$, there are three terms, $+a^2$, $-2ab$, $+3b^2$.

Each of the numbers which multiplied together form a product is called a *factor* of the product.

A factor consisting of one or more letters is called a *literal* factor.

Terms which have the same literal factors are called *similar;* terms which do not have the same literal factors are called *dissimilar.*

In $4a^2b - 3ab^2 - 10a^2b$, the terms $+4a^2b$ and $-10a^2b$ are similar; the terms $-3ab^2$ and $-10a^2b$ are dissimilar.

An algebraic expression of *one* term is called a *monomial*, of *two* terms a *binomial*, of *three* terms a *trinomial*, and of *several* terms a *polynomial*.

When a number is the product of two factors, either factor may be called the *coefficient* of the other factor.

Often the word coefficient is applied only to the factor which is expressed in numerals.

Thus, in $5\,xyz^2$, the numeral 5 is called the coefficient of xyz^2.

An *exponent* is a number placed at the right and a little above another number, called the base.

When the exponent is a positive integer, it indicates how many times the base is taken as a factor.

The exponent expresses the *power* to which the number is raised.

Thus, the 4 in a^4 expresses the *fourth power* of a.

Find a^4 when $a = 2$. Has a^4 the same value as $4\,a$, when $a = 2$?

The *absolute value* of a number is its value regardless of its sign.

Thus $+5$ and -5 have the same absolute value, 5.

ADDITION AND SUBTRACTION

2. *Addition:* If similar terms have *like* signs, find the *sum* of the absolute values and prefix the common sign.

If similar terms have *unlike* signs, find the *difference* of the absolute values and prefix the sign of the one which has the greater absolute value.

If the terms are dissimilar, the addition is indicated in the usual way.

Thus, the sum of $3\,a$ and $5\,b$ is $3\,a + 5\,b$.

To check an example in addition:

Substitute numerals for the letters and find the sum. This sum must equal the result of substituting the numerals in the answer.

3. *Subtraction:* Conceive the sign of the subtrahend to be changed, and then proceed as in addition. Check by substituting numerals for the letters.

In what other way may subtraction be checked?

WRITTEN EXERCISES

4. Find the sum of:

1. $7a - 3b + 6c$; $4a + 5b - 12c$; $-6a + 2b + 3c$.
2. $4x^2 - 3x^2y - 4xy^2$; $3x^2y - 8xy^2 - 8x^2$; $3x^2 - 6x^2y$.
3. $5a^n - 6a^nb^n - 7c^n$; $2a^n + 3a^nb^n + 8c^n$; $-5a^nb^n - 2c^n$.
4. $3(a - b) + 4(a + b)$; $7(a - b) - 5(a + b)$; $-6(a - b) + 11(a + b)$.
5. $a^4 - 3a^3 + 9a^2 - 4a + 6$; $7a^4 + 3a^3 + 2a^2 + 10a + 7$; $-4a^4 - 8a^3 - 5$.
6. From $2m - 3mn + 4n$ take $m - 3mn - 9n$.
7. From $2a^2 - 2ab + 3b^2$ take $a^2 - ab - b^2$.
8. From $4 + 3c - 8d - 9e$ take $7c + 5e - 10 - 2d$.
9. From $a^x + 2a^xb^y - 6b^y$ take $2a^x - 5a^xb^y + 8b^y$.
10. From $3(x + y) - 7(x - y)$ take $4(x - y) - 5(x + y)$.
11. From $(x - y)a - (x + y)b$ take $(x + y)a + (x - y)b$.
12. What number added to $7x^2y - 5xy^2 - 6xy$ will give $2xy^2 - 3x^2y + 4xy$?

ORDER OF FUNDAMENTAL OPERATIONS

5. The values of many algebraic expressions depend upon the order in which the operations of addition, subtraction, multiplication, and division are performed. To avoid confusion it has been found necessary to adopt certain rules, so that an expression shall always be interpreted in the same way and

represent the same value. The following rules are agreed upon:

I. A succession of multiplications and divisions shall be performed in the order in which they occur.

Thus $6 \cdot 8 \div 4 \cdot 3 = 48 \div 4 \cdot 3 = 12 \cdot 3 = 36$.

II. A succession of additions and subtractions shall be performed in the order in which they occur.

III. A succession of multiplications, divisions, additions, and subtractions shall be performed in accordance with I and II; the multiplications and divisions being performed before any additions and subtractions.

Thus, $20 + 6 \cdot 7 - 4 - 3 \cdot 5 = 20 + 42 - 4 - 15 = 43$;
$17 + 15 \div 3 - 8 \cdot 2 \div 4 = 17 + 5 - 16 \div 4 = 17 + 5 - 4 = 18$.

6. If there are parentheses, apply the preceding rules to the expressions within the parentheses first; then to the resulting expression as a whole.

The forms ( ), [ ], { } go by the general name of "parentheses" but they are designated by special names when it is necessary to distinguish one from the other.

Thus, [ ] is called a "bracket," { } a "brace," but ( ) is always called a "parenthesis."

The "vinculum" —— is also used to denote aggregation, thus $\overline{a+b} \div \overline{a-b}$.

ORAL EXERCISES

7. Simplify:

1. $16 - 2 + 14$.
2. $3x - 4y + 8x + 3y$.
3. $2a - 7a - 3a + 8a$.
4. $10 - 5m - 15 + 20m$.
5. $5 - (7 - 2) + [8 + 3]$.
6. $25 \div 5 + 15 + 40$.
7. $28 \div 7 \cdot 2 + 5$.
8. $\frac{1}{3}x - \frac{1}{2}y - \frac{1}{2}x + \frac{4}{5}y$.
9. $15 \cdot 3 \div 5 + 10 - 4$.
10. $24 + 5 \div 10 - 7 - (15 \div 2)$.
11. $48 + 3(18 \div 9 \cdot 3) \div 9$.
12. $(12a - 7a) - (13a - 20a)$.

MULTIPLICATION

8. *To multiply one term by another:* Multiply the numerical coefficients; annex the letters, giving to each letter in the product an exponent equal to the sum of its exponents in the two factors.

Law of signs: Like signs give plus, unlike signs give minus.

The *degree* of a term is the sum of the exponents of the literal factors.

Thus $4xy^2z^3$ is of the sixth degree.

An expression is *homogeneous*, if all its terms are of the same degree.

In $x^4 + 2x^2y^2 - 5xy^3$ every term is of the fourth degree; the expression is homogeneous.

To multiply a polynomial by a polynomial: Arrange the terms in both polynomials according to the ascending or descending powers of some letter; multiply each term of the multiplicand by each term of the multiplier and add the partial products.

Multiplication may be performed by *detached* coefficients.

For example, $(2x^4 - 7x^3 - 5x^2 + 6x - 3)(6x^2 - 4x + 5)$.

$$\begin{array}{l}
2 - 7 - 5 + 6 - 3 \\
6 - 4 + 5 \\
\hline
12 - 42 - 30 + 36 - 18 \\
\quad\;\; - \;\; 8 + 28 + 20 - 24 + 12 \\
\qquad\qquad\quad 10 - 35 - 25 + 30 - 15 \\
\hline
12 - 50 + \;\; 8 + 21 - 67 + 42 - 15
\end{array}$$

The product is $12x^6 - 50x^5 + 8x^4 + 21x^3 - 67x^2 + 42x - 15$.

If any powers of x are lacking in either polynomial, the terms in question must be represented by zeros.

Thus, in multiplying $x^3 + 2x - 4$ by $x^4 + x^2 + 1$, we must write $1 + 0 + 2 - 4$ and $1 + 0 + 1 + 0 + 1$.

Check an example in multiplication by substituting numerals for the letters.

DIVISION

9. *To divide one term by another:* Divide the numerical coefficients; annex the letters, giving to each letter in the quotient an exponent equal to its exponent in the dividend minus its exponent in the divisor. Observe the law of signs: Like signs give plus, unlike signs give minus.

To divide one polynomial by another:

I. Arrange the terms.

II. Divide the first term of the dividend by the first term of the divisor and obtain the first term in the quotient.

III. Multiply the divisor by this term in the quotient.

IV. Subtract the product from the dividend.

V. Treat the remainder as a new dividend and proceed as before.

VI. Keep each new dividend arranged in the same order as the first dividend.

Divide $14x^4 - 27ax^3 + 21a^2x^2 - 32a^4 + 12a^3x$ by $2x^2 - 3ax + 4a^2$.

$$
\begin{array}{rrrrr|l}
14x^4 & -27ax^3 & +21a^2x^2 & +12a^3x & -32a^4 & 2x^2 - 3ax + 4a^2 \\
14x^4 & -21ax^3 & +28a^2x^2 & & & 7x^2 - 3ax - 8a^2 \\
\hline
 & -6ax^3 & -7a^2x^2 & +12a^3x & -32a^4 & \\
 & -6ax^3 & +9a^2x^2 & -12a^3x & & \\
\hline
 & & -16a^2x^2 & +24a^3x & -32a^4 & \\
 & & -16a^2x^2 & +24a^3x & -32a^4 & \\
\hline
 & & & & 0 &
\end{array}
$$

By detached coefficients this division may be performed as follows:

$$
\begin{array}{rrrrr|l}
14 & -27 & +21 & +12 & -32 & 2 - 3 + 4 \\
14 & -21 & +28 & & & 7 - 3 - 8 \\
\hline
 & -6 & -7 & +12 & -32 & \\
 & -6 & +9 & -12 & & \\
\hline
 & & -16 & +24 & -32 & \\
 & & -16 & +24 & -32 & \\
\hline
 & & & & 0 &
\end{array}
$$

The quotient is $7x^2 - 3ax - 8a^2$.

Division may be checked by substituting numerals for the letters, care being taken to avoid a zero divisor.

ORAL EXERCISES

10. 1. In what other way may examples in multiplication and division be checked?

2. If there is a remainder in a division, what must be done with it in checking?

3. If the multiplicand and the multiplier are arranged according to the descending powers of some letter, how will the product be arranged?

4. If the multiplicand and the multiplier are homogeneous, what will be true of the product? Of what degree will it be?

WRITTEN EXERCISES

11. Expand and check:

1. $(7a + 2a^2 - 4 + 2a^3)(3a - 3 + 4a^2)$.
2. $(x^{2a} - 2x^a y^b + y^{2b})(x^a - y^b)$.

Divide and check:

3. $a^3 - 3a^2b + 3ab^2 - b^3$ by $a - b$.
4. $x^6 - y^6$ by $x - y$, and by $x + y$.
5. $12x^{a+4} + 20x^{a+3} - 107x^{a+2} + 37x^{a+1} + 33x^a$ by $6x^3 - 11x^2$.
6. $m^3 + n^3 + p^3 - 3mnp$ by $m + n + p$.
7. $a^3 - b^3 + c^3 + 3abc$ by $a^2 + b^2 + c^2 + ab - ac + bc$.
8. $y^5 - \frac{5}{4}y^4 + \frac{29}{24}y^3 - \frac{31}{24}y^2 + \frac{5}{6}y - \frac{1}{4}$ by $y - \frac{3}{4}$.
9. $c^6 - c^5 + \frac{1}{2}c^4 + \frac{5}{24}c^3 - \frac{13}{6}c^2 + \frac{13}{12}c - \frac{1}{2}$ by $c^4 - \frac{1}{2}c^2 + \frac{1}{3}c - 2$.

PARENTHESES

12. I. A parenthesis preceded by a + sign may be removed without changing the signs of the terms within the parenthesis.

II. A parenthesis preceded by a − sign may be removed, provided that the sign of each term within the parenthesis be changed.

WRITTEN EXERCISES

13. Simplify:

1. $18 - (7 - 2) - 3.$
2. $5a + (6a - 4a) - (9a - 2a).$
3. $8x - [2x + (3x - y) + 7] + [5x - \overline{2y + 3}].$
4. $-7m + [2n - \{4m - (3m - 5n) - 8n\} + 9m].$
5. $10c - \{4d - (9c - 5d)\} - \{(2c - d) - (4c + 7d)\}.$
6. $(x + 2)(x - 3) - 5(x^2 - 3x + 2) - 10(x + 3).$
7. $\{5a^2 - 3[-5 + (a + 2)(a - 5) - 7]\} - a(a - 1)^2.$
8. $(y - 3)^3 - y^2(y - 3)^2 + (-y^2 + 7y + 8).$
9. $(a + b)(a - b) - (a - b)^2 + (a + b)^2.$
10. $2\{-4 - (7 - 3) + (-8 - 2) - 5(9 - 3 + 5)\} + 11.$
11. $m - \{-[-(1 - m) - 1] - m\} - \{m - (5 - 4m) - (4 + m)\}.$

14. Terms may be inclosed in a parenthesis by reversing the rules for removing parentheses.

When terms are inclosed in a parenthesis preceded by a + *sign*, the signs of the terms *are not changed.*

When terms are inclosed in a parenthesis preceded by a − *sign*, the signs of *all* the terms *are changed.*

WRITTEN EXERCISES

15. Collect the coefficients of x in a parenthesis preceded by a + sign, and the coefficients of y in a parenthesis preceded by a − sign:

1. $ax - ay + bx - by.$
2. $mx + nx + my - ny.$
3. $abx + cdy - cdx - aby.$
4. $2x - 3y - dx + dy - fx.$
5. $py - qx - px + qy.$
6. $-7ax + 3by + 4bx - 5ay.$

REVIEW EXERCISES

16. If $A = 2a^2 - 4ab + 7b^3$, $B = -5a^2 + 2ab - 4b^2$,
$C = 3a^2 + 10ab - 6b^2$, $D = 4a^2 - 9b^2 + 3ab$,

find the values of:

1. $A + B + C + D$.
2. $A - B + C - D$.
3. $-A + B - C + D$.
4. $B - A + C - D$.
5. $C - B + A - D$.
6. $D + A - C - B$.

If $a = 1$, $b = 3$, $c = -2$, $d = 4$, $n = 2$, find the numerical values of:

7. $\dfrac{2a - 3c + 4d}{7c - 2d}$.
8. $\dfrac{a}{b} + \dfrac{c}{d}$.
9. $\dfrac{a^2 + b^2}{c^2 + d^2}$.
10. $\dfrac{a^n - b^n}{c^n + d^n}$.
11. $5a \div (b - c) + d - n$.
12. $3a^2 - 2a^n - 2b^n$.
13. $(a + b)^2 - (c - d)^3$.
14. $(c + d)(c - d) + \dfrac{c + d}{c - d}$.
15. $2d \div 3c \cdot dn - bc \div d$.
16. $\sqrt{b^2} + \sqrt{d^2}$.

Find the sum of:

17. $(a + b)x - (a - b)y$ and $(a - b)x + (a + b)y$.
18. $\frac{1}{2}a - \frac{2}{3}b + \frac{1}{5}c$; $\frac{1}{3}a + \frac{1}{6}b - \frac{3}{10}c$; and $\frac{5}{6}a + \frac{1}{3}b - 1\frac{1}{2}c$.
19. $\frac{2}{5}x - \frac{3}{7}y + \frac{1}{6}z$; $\frac{2}{3}x + \frac{4}{9}y - \frac{1}{2}z$; and $\frac{1}{5}x - \frac{5}{9}y + \frac{2}{3}z$.
20. $[(a + c)x + (b + c)y] - [(a - c)x - (b - c)y]$.
21. $[5(x + y) - 7(x - y)] - [-2(x + y) - 3(x - y)]$.

Simplify:

22. $x^2 - (y^2 - z^2) - [y^2 - (z^2 - x^2)] + [z^2 - (y^2 - x^2)]$.
23. $x - 2z - \{3y - w - [3x - (5z - y - 7w)] - 5z + 4x\}$.
24. $(x^a + 2x^{a-1} - 3x^{a-2} - 1)(x - 1)$.

25. $(a^{n+1} - 4\,a^n + 5\,a^{n-1} + a^{n-2})(a + 1)$.

26. $(a^n + b^n)(a^n - b^n)$.

27. $(\frac{1}{2}\,a^2 - \frac{2}{3}\,ab + \frac{1}{5}\,b^2)(\frac{2}{3}\,a - \frac{7}{8}\,b)$.

28. $(a^2b^{m+1} - 5\,a^3b^m + 4\,a^nb^{m-1}) \div 2\,a^nb^{2m-3}$.

29. $(x^3y^a - x^{a+1}y^{a+1} + x^{2-a}y^{3a-a}) \div x^{a-5}y^{a-1}$.

30. $(a^{4x} + a^{2x} + 1) \div (a^{2x} - a^x + 1)$.

31. $(9\,a^{2n} - 25\,b^{4n}) \div (3\,a^n + 5\,b^{2n})$

32. $(\frac{1}{4}\,x^3 - \frac{1}{2}\,x^2y + \frac{25}{72}\,xy^2 - \frac{1}{6}\,y^3) \div (\frac{1}{2}\,x - \frac{2}{3}\,y)$.

TYPE FORMS IN MULTIPLICATION

17. 1. $(a \pm b)^2 = a^2 \pm 2\,ab + b^2$.

Take the upper signs together, and the lower signs together.

2. $(a + b - c)^2 = a^2 + b^2 + c^2 + 2\,ab - 2\,ac - 2\,bc$.

3. $(a + b)(a - b) = a^2 - b^2$.

4. $(x + a)(x + b) = x^2 + (a + b)x + ab$.

5. $(ax + b)(cx + d) = acx^2 + (ad + bc)x + bd$.

6. $(a \pm b)^3 = a^3 \pm 3\,a^2b + 3\,ab^2 \pm b^3$.

ORAL EXERCISES

18. Write, by inspection, the values of:

1. $(a + 2)^2$.
2. $(b - 3)^2$.
3. $(2\,x - 1)^2$.
4. $(3\,a - 4)^2$.
5. $(a^2 + b)^2$.
6. $(c^3 - d^3)^2$.
7. $(c^2 + 7\,d^2)^2$.
8. $(5\,x - 3\,y^2)^2$.
9. $(a^n - b^n)^2$.
10. $(x^3 + b^2)^2$.
11. $(m^3 - 2\,n^2)^2$.
12. $(8\,a - 5\,x^n)^2$.
13. $(x - y + z)^2$.
14. $(2\,x - y - z)^2$.
15. $(5\,x - 2\,y + z)^2$.
16. $(x - 3\,y - 4)^2$.
17. $(x - 7)(x + 3)$.
18. $(x + 4)(x + 9)$.
19. $(x + 20)(x - 15)$.
20. $(x - 8)(x - 5)$.
21. $(a + 4\,c)(a - 3\,c)$.
22. $(2\,a + c)(3\,a - d)$.

23. $(3x+4)(2x-7)$.

24. $(5b-6c)(4b+3c)$.

25. $(x-y)(a-b)$.

26. $(3y-4)(3y+8)$.

27. $(a^2+c)(a^2-c)$.

28. $(2a+b)(2a-b)$.

29. $(5x^2-y^3)(y^3+5x^2)$.

30. $(x^a+y^b)(x^a-y^b)$.

31. $(a-2c)(a-7c)$.

32. $(a^2+2b)(a^2+3b)$.

33. $(a^3-9c)(a^3+8c)$.

34. $(5a-4b)^3$.

35. $(x^2+y)^3$.

36. $(3x-4)^3$.

37. $(x-y)^3$.

38. $(2a+1)^3$.

Complete the squares:

39. $4x^2 \pm 12x + ?$

40. $9a^2 \pm 24a + ?$

41. $? \pm 18bc + 81c^2$.

42. $4m^2 \pm ? + 9n^2$.

43. $a^{2x} \pm ? + 36b^{2x}$.

44. $25x^2 - 20 + ?$

EQUATIONS

19. An *equation* expresses an equality.

In other words, an *equation* is a statement that two expressions stand for the same number.

$2x+6=7x-4$, and $(a+b)^2=a^2+2ab+b^2$ are equations. The first is an equation of *condition*, because it is true only under the condition that $x=2$. The second is an equation of *identity*, because it is true whatever values a and b may have.

In this chapter we deal with equations of condition. The *root* of an equation, containing one unknown, is a number which, when substituted for the unknown, "satisfies" the equation by reducing both sides to identical numbers.

An equation of the form

$$ax^n + bx^{n-1} + \ldots + l = 0,$$

where n is a positive integer, is said to be of the nth degree.

The *degree* of any equation is indicated by the exponent of the highest power of the unknown.

Thus, $x - 7 = 2$, is an equation of the 1st degree, called *linear*.
$x^2 - 3x = -2$, is an equation of the 2d degree, called *quadratic*.
$x^3 - 6x^2 + 11x = 6$, is an equation of the 3d degree, called *cubic*.
$x^4 - 1 = 0$, is an equation of the 4th degree, called *quartic*.
etc.

20. The equilibrium of a balance is not disturbed so long as like changes in the weights are made simultaneously on both sides. So in equations, we may *add* the same number to both sides, or *subtract* the same number from both sides, or we may *multiply* or *divide* both sides by the same number (except division by zero).

The equality is maintained during all these changes.

Solve $x(x+4) = x^2 - 3x + 5.$

Remove the parenthesis, $x^2 + 4x = x^2 - 3x + 5.$
Subtract x^2 from both sides, $4x = -3x + 5.$
Add $3x$ to both sides, $7x = 5.$
Divide both sides by 7, $x = \frac{5}{7}.$

Check: In the given equation, substitute $\frac{5}{7}$ for x,

$$\tfrac{5}{7}(\tfrac{5}{7} + 4) = \tfrac{25}{49} - \tfrac{15}{7} + 5.$$
$$\tfrac{25}{49} + \tfrac{20}{7} = \tfrac{25}{49} - \tfrac{15}{7} + 5.$$
$$\tfrac{20}{7} = -\tfrac{15}{7} + 5.$$
$$\tfrac{35}{7} = 5.$$

WRITTEN EXERCISES

21. Solve and check:

1. $(x-2)(x-3) = x(x+4).$
2. $(2y+3)(3y-2) = 6y^2 + 2(2y + 4\tfrac{1}{2}).$
3. $(m-4)^2 + (m+4)^2 = m(2m+1).$
4. $(r+1)^3 - r(r-1) = r^2(r+2) + 5r - 1.$
5. $y(y+1) + (y+1)(y+2) = (y+2)(y+3) + y(y+4) - 9.$
6. $t - [4 + \{4 - (4+t)\}] = 0.$
7. $6x - (4x - 8) - \{5 - 3x - (7x - 4)\} = 11.$

8. $s + 3 - [s - 9 - 3\{9 - 4(6 - s) - s\}] = 3.$

9. $20(1 - x) - 3(x - 6) - 3[x + 8 - 4\{8 - 3(1 - x)\}] = -6.$

10. $(x + 8)(x - 5)(x + 6) = (x + 2)(x + 3)(x + 4).$

Solve each of the following equations for each letter in terms of the others:

11. $A = \frac{1}{2}ab.$

12. $S = \frac{1}{2}a(b + b').$

13. $S = \frac{n}{2}(a + l).$

14. $l_1W_1 = l_2W_2.$

15. $C = \frac{5}{9}(F - 32).$

SIMULTANEOUS LINEAR EQUATIONS

22. If a linear equation contains but one unknown, one value may be found for that unknown.

If a linear equation contains two unknowns, an unlimited number of simultaneous values may be found for them. For that reason the equation is called *indeterminate.*

In $x + y = 3$, we have $x = 1$, $y = 2$; $x = 2$, $y = 1$; $x = -2$, $y = 5$; $x = 0$, $y = 3$; $x = 3$, $y = 0$; etc.

If these pairs of values are plotted, the points thus obtained will be seen to lie on a straight line, as in Fig. 1; hence the name *linear* equation.

Two points determine the position of a straight line. The graph of a linear equation in two unknowns is most easily made by locating the points at which the line crosses the coördinate axes.

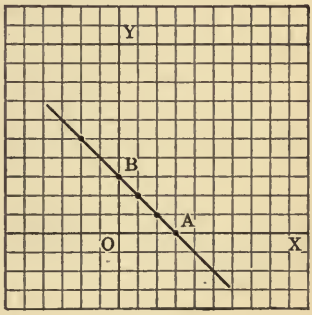

FIG. 1.

It was seen above that in the equation $x + y = 3$, $x = 3$ when $y = 0$, and $x = 0$ when $y = 3$. In Fig. 1,

$x = 3$, $y = 0$ locates the point A.
$x = 0$, $y = 3$ locates the point B.

The two points A and B determine the line.

EXERCISES

23. Make graphs of the following equations:

1. $x+y=7.$
2. $x-y=2.$
3. $x-15=3y.$
4. $3x-4y=12.$
5. $6-2y-x=0.$
6. $4x-9y=36.$
7. $5x=10-2y.$
8. $x=y.$
9. $x-3y=0.$
10. $3x-y=6.$
11. $y=0.$
12. $x=0.$
13. $x=3.$
14. $y=-4.$
15. $x=-2.$

24. If the graphs of two linear equations in x and y intersect, the values of x and y at the point of intersection satisfy both equations; this value of x and of y, called the coördinates of the point, constitute one solution.

Since two straight lines cannot intersect in more than one point, two linear equations in x and y cannot have more than one solution.

Two linear equations in x and y, representing lines which intersect in one point, are called *independent.*

WRITTEN EXERCISES

25. Make graphs of the following equations and determine, from the graph, the coördinates of the point of intersection. Check, by substituting the coördinates in the equations.

We are not able to draw figures that are absolutely accurate. Hence, solutions obtained from graphs are usually only *approximations* to the true values.

1. $x+y=5,$ $x-y=1.$
2. $3x-y=7,$ $2x-y=5.$
3. $x+3y=8,$ $2x+y=1.$
4. $x=3,$ $2x-3y=9.$
5. $y=-2,$ $5x-3y=16.$
6. $x=y-5,$ $6y=x+10.$
7. $3x=2y,$ $x+4=7y.$
8. $x-y=0,$ $3x+2y=5.$
9. $x+1=0,$ $y-x=6.$

26. The two linear equations,

$$x + y = 5,$$
$$2x + 2y = 10,$$

can be satisfied by the same values of x and y; for example, $x = 1$ and $y = 4$, or $x = 3$ and $y = 2$, or $x = 5$ and $y = 0$, etc. Moreover, by dividing both sides of the second equation by 2, the second equation can be reduced to the first. The two equations are therefore not independent; they are really different forms of one and the same equation and represent one and the same straight line. They are called *equivalent* equations.

27. The two linear equations,

$$x - y = 5,$$
$$x - y = 7,$$

are called *inconsistent*, for the reason that no finite value of x and of y can be found, such that $x - y$ is equal to 5 and also equal to 7; no pair of finite values of x and y satisfy both equations. This is geometrically evident from the fact that the graphs of the two equations are *parallel lines.*

Two linear equations in two variables x and y belong therefore to one of three groups:

1. The two equations are *independent* and represent two lines which intersect in one point, or
2. The two equations are *inconsistent* and represent two lines which do not intersect (are parallel), or
3. The two equations are *equivalent* and represent one and the same straight line.

WRITTEN EXERCISES

28. Make graphs of the following equations:

1. $3x - 6y = 8,$
   $x - 2y = 5.$
2. $5y - 2x = 4,$
   $4x - 10y = -8.$
3. $2y + 3x = 0,$
   $9x + 6y = 15.$
4. $2x - 4y = -10,$
   $x - 2y = -5.$

5. $x - 3y = 2x - 7,$
$3x + 2y = 4x + 5y - 7.$

6. $5x - 7y = 11,$
$10x - 14y = 20.$

7. $x = 3,$
$y = -2.$

8. $x + y = 0,$
$x - y = 0.$

9. $x + 4 = 0,$
$y - 1 = 0.$

10. $x = 0,$
$y = 0.$

11. Can you tell, before making the graphs, whether two equations are equivalent, or not? Inconsistent, or not?

12. Can you tell whether the graphs will be parallel, or not?

13. Can you tell whether the graphs will go through the origin, or not?

29. Systems of independent linear equations may be solved by a process called *elimination.*

The elimination may be accomplished in one of three ways:

I. *Addition or subtraction.* — Make the coefficients of one variable alike in both equations and then either add or subtract the sides of the equations.

II. *Substitution.* — Find the value of one variable in terms of the other in *one* equation, and substitute that value in the *other* equation.

III. *Comparison.* — Find the value of one variable in terms of the other in *both* equations, then equate these values and solve the resulting equation.

EXERCISES

30. 1. Solve by addition or subtraction:

$$5x + 3y = 69, \qquad (1)$$
$$4x - 7y = -20. \qquad (2)$$

Solution. Multiply (1) by 4, (2) by 5, $20x + 12y = 276, \qquad (3)$
$20x - 35y = -100. \qquad (4)$

Subtract (4) from (3), $47y = 376,$
$y = 8.$

Substitute in (1), $x = 9.$

Check: Substitute in (1) and (2), $45 + 24 = 69,$
$36 - 56 = -20.$

2. Solve by substitution,

$$\frac{x}{5}+\frac{y}{6}=4, \qquad (1)$$

$$\frac{x}{7}-\frac{y}{8}=-\frac{1}{14}. \qquad (2)$$

Solution. From (1),

$$x=20-\frac{5y}{6}.$$

Substitute in (2),

$$\frac{20-\frac{5y}{6}}{7}-\frac{y}{8}=-\frac{1}{14},$$

$$160-\frac{20y}{3}-7y=-4,$$

$$\frac{41y}{3}=164,$$

$$\frac{y}{3}=4,$$

$$y=12.$$

Substitute in (1),

$$\frac{x}{5}+2=4,$$

$$x=10.$$

3. Solve by comparison,

$$3x-13y=1, \qquad (1)$$

$$15x-y=9\tfrac{12}{13}. \qquad (2)$$

Solution. From (1),

$$y=\frac{3x-1}{13}. \qquad (3)$$

From (2),

$$y=15x-9\tfrac{12}{13}. \qquad (4)$$

Compare (3) and (4),

$$\frac{3x-1}{13}=15x-9\tfrac{12}{13},$$

$$3x-1=195x-129,$$

$$192x=128,$$

$$x=\tfrac{2}{3}.$$

Substitute in (3),

$$y=\tfrac{1}{13}.$$

Solve and check:

4. $12x+11y=12,$
$42x+22y=40\tfrac{1}{2}.$

5. $\frac{x}{3}=10-\frac{9y}{5},$
$\frac{3y}{2}-6=\frac{x}{2}.$

6. $\frac{x}{3}+5=\frac{2y}{3},$
$y-x=\frac{x}{3}.$

7. $\frac{x}{3}=y+1,$
$\frac{x+y}{3}+4=x-3.$

8. $2x + \frac{y}{3} = x + 12,$

$y - x + 20 = \frac{x + 40}{2}.$

9. $\frac{2(x + y)}{9} = \frac{x + 5}{3},$

$\frac{y + 20}{4} = \frac{12x - 5y}{20}.$

PROBLEMS

31. 1. The sum of two numbers is 180 and their difference is 12. Find the numbers.

2. One number is 27 more than another. Their sum is 143. What are the numbers?

3. Of two numbers, twice the greater exceeds three times the smaller by 4; but the sum of the greater and twice the smaller is 44. Find the numbers.

4. Two pounds of sugar and three pounds of flour cost 19¢, and three pounds of sugar and five pounds of flour cost 30¢. What is the price of each per pound?

5. Three yards of velvet and five yards of silk cost me $15.75. If I had bought two yards more of velvet and two yards less of silk, my bill would have been $18.25. What is the cost of the velvet and silk per yard?

6. In five hours A can walk 1 mi. more than B can walk in 6 hours; in seven hours A can walk 3 mi. more than B can walk in 8 hours. How many miles an hour can each walk?

7. A, B, C, D, together have $300. A has $10 more than C; B has $5 less than half as much as D; and A and B have together $5 less than twice as much as C. How much has each?

8. A number consists of two digits whose sum is 12. If three times the sum of the digits be subtracted from the number, the digits will be reversed. What is the number?

9. The digits of a three-digit number will be reversed, if 396 be added to the number. The units' digit is 3 times the hundreds', and if double the tens' digit be increased by 6, the result will equal the units' digit. Find the number.

10. In a purse there are 3 times as many quarters as nickels, and twice as many half dollars as nickels. The total value of the coins is \$3.60. How many of each kind are there?

11. Some boys bought a boat and found upon paying for it that if there had been 2 more of them, each would have paid a dollar less; but if there had been 2 fewer, each would have had to pay \$$1\frac{1}{2}$ more. How much did the boat cost?

12. A rectangle is four times as long as it is wide. If it were 3 in. shorter and 2 in. wider, its area would be increased 15 sq. in. Find its dimensions.

13. It costs as much to sod a square piece of ground at 25¢ a square yard as it does to fence it at \$1 a yard. How long is the side of the square?

14. A walk is laid 3 ft. wide around a rectangular court, which is 15 ft. longer than it is wide. The area of the walk is 960 sq. ft. Find the dimensions of the court.

15. A man invests part of \$5480 at 5 % and the remainder at 4 %. The total annual income is \$244. How many dollars has he in each investment?

16. The yearly income from a 5 % investment is \$97.70 more than that from a 6 % investment. The sum of the two investments is \$6420. How much is invested at 6 %?

17. A boy weighing 100 lb. is 6 ft. from the fulcrum of a seesaw. He balances a boy who is 8 ft. from the fulcrum. What is the weight of the second boy?

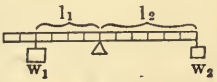

FIG. 2.

By careful measurement it has been ascertained that, to maintain a balance, the lengths of the arms of a lever and the corresponding weights must conform to the law, $l_1w_1 = l_2w_2$; that is, the product of one weight and its distance from the fulcrum is equal to the product of the other weight and its distance from the fulcrum.

18. Two boys together weigh 170 lb. They balance when one is 8 ft. and the other 9 ft. from the fulcrum. How much does each weigh?

19. If the lever is 8 ft. long, how far from the fulcrum will two weights, 30 lb. and 50 lb., have to be in order to balance?

20. A weight of 180 lb. is carried between two men by means of a pole. One man is 5 ft. from the weight, the other is 4 ft. How many lb. does each man lift?

POSITIVE INTEGRAL EXPONENTS

32. In the expression a^n we say *a is raised to the nth power.* When n is a positive integer:

I. The nth power of a is the product obtained by using a n times as a factor.

$$a \cdot a \cdot a \cdots \text{ to } n \text{ factors} = a^n.$$

II. The nth power of a^m is equal to $a^m \cdot a^m \cdot a^m \cdots$ to n factors $= a^{mn}$.

Hence $$(a^m)^n = a^{mn}.$$

III. The nth power of $a^m b^p$ is equal to $(a^m \cdot a^m \cdot a^m \cdots \text{ to } n \text{ factors})(b^p \cdot b^p \cdot b^p \cdots \text{ to } n \text{ factors}) = a^{mn} \cdot b^{pn} = a^{mn} b^{pn}$.

Hence $$(a^m b^p)^n = a^{mn} b^{pn}.$$

IV. The nth power of $\frac{a^m}{b^p}$ is equal to

$$\frac{a^m \cdot a^m \cdot a^m \cdots \text{ to } n \text{ factors}}{b^p \cdot b^p \cdot b^p \cdots \text{ to } n \text{ factors}} = \frac{a^{mn}}{b^{pn}}.$$

Hence, $$\left(\frac{a^m}{b^p}\right)^n = \frac{a^{mn}}{b^{pn}}.$$

33. From the law of the signs in multiplication:

1. An *even* power of any real number is *positive.*

2. An *odd* power of any real number has the same sign as the number itself.

EXERCISES

34. Write the values of:

1. $(2a^2b)^3$.
2. $(-7x^3y)^2$.
3. $(-2ab^3)^5$.
4. $\left(\frac{3a^2b^2}{4c^5}\right)^2$.
5. $\left(\frac{-5axy^3}{2bc}\right)^3$.
6. $(-6a^nb^n)^2$.
7. $(x^4y^3z^2)^n$.
8. $(a^{m+n}b^{m-n})^r$.
9. $(-\frac{3}{4}a^5)^4$.
10. $\left(\frac{1}{2y^3}\right)^3$.
11. $\left(-\frac{a^3}{3}\right)^7$.
12. $\left(\frac{-3a^2b}{2a^3b^2}\right)^5$.
13. $(2a^2bc^3)^x$.
14. $-(-a^2b^3c^4)^2$.
15. $-[-(a^3)^3]^3$.
16. $\left(\frac{a^mb^mc^m}{-a^nb^nc^n}\right)^4$.
17. $\left(\frac{-x^2y^3z^5}{a^2b^2c^4}\right)^3$.
18. $(r^2s^4t^2v^5)^6$.
19. $(-\frac{2}{3}a^2b^n)^{3n}$.
20. $(6x^3y^2z)^{2r}$.

MEANING OF FRACTIONAL EXPONENTS

35. When an exponent n is a positive integer, we know that $a^n = a \cdot a \cdot a \cdots$ to n factors. That is, the exponent n indicates that a is taken n times as a factor.

What is the meaning of $a^{\frac{1}{2}}$? It would be absurd to say that $a^{\frac{1}{2}}$ signifies a taken $\frac{1}{2}$ times as a factor. A number can be taken as a factor only a whole number of times.

For the purpose of attaching a meaning to $a^{\frac{1}{2}}$ we stipulate that we shall be able to multiply $a^{\frac{1}{2}}$ by $a^{\frac{1}{2}}$ according to the same rule by which a^4 is multiplied by a^5. The product of a^4 and a^5 is found by *adding the exponents;* that is, $a^4 \cdot a^5 = a^9$.

If we add the exponents in the multiplication of $a^{\frac{1}{2}}$ by $a^{\frac{1}{2}}$, we obtain,

$$a^{\frac{1}{2}} \cdot a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a.$$

It is seen that $a^{\frac{1}{2}}$ is *one of the two equal factors of* a, or the *square root* of a.

Hence $a^{\frac{1}{2}}$ is another way of writing the square root of a. Likewise, the meaning of $a^{\frac{3}{4}}$ is found by taking $a^{\frac{3}{4}}$ four times as a factor, thus,

$$a^{\frac{3}{4}} \cdot a^{\frac{3}{4}} \cdot a^{\frac{3}{4}} \cdot a^{\frac{3}{4}} = a^3.$$

Hence $a^{\frac{3}{4}}$ means *one of the four equal factors of* a^3 or the *fourth root* of a^3.

In general, to find a meaning for $a^{\frac{p}{q}}$, where p and q are positive integers, we take $a^{\frac{p}{q}}$ as a factor q times, and add the exponents. We obtain,

$$a^{\frac{p}{q}} \cdot a^{\frac{p}{q}} \cdot a^{\frac{p}{q}} \cdots \text{ to } q \text{ factors} = a^{\frac{p}{q}+\frac{p}{q}+\frac{p}{q}+\cdots \text{ to } q \text{ terms}} = a^{\frac{pq}{q}} = a^p.$$

It is seen that $a^{\frac{p}{q}}$ is one of the q equal factors of a^p. Hence $a^{\frac{p}{q}}$ means the qth root of a^p. Thus,

The numerator of a positive fractional exponent indicates the power of the base and the denominator indicates the root of that power.

In finding the value of $27^{\frac{2}{3}}$, we may square 27, which gives 729, and then take the cube root of 729, which is 9. Or we may take the cube root of 27, which is 3, and square 3, getting 9 as the result, that is,

$$27^{\frac{2}{3}} = \sqrt[3]{27^2} = (\sqrt[3]{27})^2.$$

In general, we have,

$$a^{\frac{p}{q}} = \sqrt[q]{a^p} = (\sqrt[q]{a})^p.$$

All irrational numbers which are expressed by the use of radical signs can be expressed by the use of fractional exponents. In fact, the simplification of expressions usually can be effected more easily by the latter. Thus,

$$\sqrt[4]{16\,a^6b^{12}c^4} = 16^{\frac{1}{4}}a^{\frac{6}{4}}b^{\frac{12}{4}}c^{\frac{4}{4}} = 2\,a^{\frac{3}{2}}b^3c.$$

MEANING OF A ZERO EXPONENT

36. We proceed to assign a meaning to a^0, where a is not itself zero. For simplicity we stipulate, as before, that in the multiplication of two powers having equal bases, we find the product *by adding the exponents.*

Accordingly, $$a^m \cdot a^0 = a^{m+0} = a^m.$$

Divide both sides by a^m, $$\frac{a^m \cdot a^0}{a^m} = \frac{a^m}{a^m}.$$

Simplify, $$a^0 = 1.$$

Therefore, *any number, except* 0, *with a zero exponent is equal to* 1.

MEANING OF NEGATIVE EXPONENTS

37. Let m represent a positive integer or a positive fraction. What is the meaning of a^{-m}, where a is not zero? Working on the assumption (§ 35 and § 36) that in multiplication we are permitted to *add the exponents,* we obtain,

$$a^m \cdot a^{-m} = a^{m-m} = a^0.$$

Since $a^0 = 1$, $$a^m \cdot a^{-m} = 1.$$

Divide by a^m, $$a^{-m} = \frac{1}{a^m}.$$

Therefore, *any number with a negative exponent is equal to* 1 *divided by that number with a positive exponent.*

Thus, $$\frac{a^{-3}b^4}{a^{-4}b^{-2}c} = \frac{\frac{1}{a^3} \cdot b^4}{\frac{1}{a^4} \cdot \frac{1}{b^2} \cdot c} = \frac{\frac{b^4}{a^3}}{\frac{c}{a^4b^2}} = \frac{b^4 \cdot a^4b^2}{a^3 \cdot c} = \frac{ab^6}{c}.$$

Therefore, a *factor* may be moved from the numerator to the denominator of a fraction, or from the denominator to the numerator, *provided the sign of its exponent be changed.*

Care must be taken to transfer only *factors*. In $\frac{a^{-2}+b}{c}$, a^{-2} is *not* a factor of the numerator. $\frac{a^{-2}+b}{c}$ is not equal to $\frac{1+b}{a^2c}$. Since $a^{-2}=\frac{1}{a^2}$, it follows that $\frac{a^{-2}+b}{c}=\frac{\frac{1}{a^2}+b}{c}=\frac{1+a^2b}{a^2c}$.

PRINCIPAL ROOTS

38. Since $6 \cdot 6 = +36$, and $(-6)(-6) = +36$, it follows that both $+6$ and -6 are square roots of 36. The two square roots of 36 are usually written in the form ± 6. Similarly the two square roots of a^2 are written $\pm a$.

The positive square root of a number is called the *principal square root.* Unless otherwise stated, we shall consider only the principal square root, and write $\sqrt{36}=6$. When $\sqrt{\ }$ has no sign before it, or has the $+$ sign before it, the principal square root is always understood.

When we write $-\sqrt{\ }$ we mean the negative square root.

Thus, $\sqrt{36}$ or $(36)^{\frac{1}{2}}$ stands for $+6$, $-\sqrt{36}$ or $-(36)^{\frac{1}{2}}$ stands for -6, $\pm\sqrt{36}=\pm(36)^{\frac{1}{2}}=\pm 6$.

It can be shown that there are 3 different cube roots of a number, four different fourth roots of a number, and, in general, n different nth roots of a number.

For example, there are three different cube roots of b^3; namely, b, $\frac{-1+\sqrt{-3}}{2}b$, $\frac{-1-\sqrt{-3}}{2}b$. The fractional coefficients in the last two roots involve the imaginary number $\sqrt{-3}$. Only one of the three roots is real, namely, the root b. We call b the *principal root.*

In elementary algebra the principal root is the only root usually considered.

I. If a is a *positive* number, then the *principal nth root of a* is its positive value; we designate it by $a^{\frac{1}{n}}$ or $\sqrt[n]{a}$. When n is an odd number, then this principal root is the only real root.

When n is *even* there is still another real root, namely, $-a^{\frac{1}{n}}$ or $-\sqrt[n]{a}$.

Thus, the principal cube root of 64 is $(64)^{\frac{1}{3}}$ or $\sqrt[3]{64} = 4$; this is the only cube root of 64 that is real.

The principal fourth root of 81 is $(81)^{\frac{1}{4}}$ or $\sqrt[4]{81} = 3$. There is another real fourth root of 81; namely, -3, for we see that $(-3)(-3)(-3)(-3) = 81$.

II. If a is a *negative* number, and n is an *odd* integer, there is no positive root; there is a negative root and that is taken as the *principal nth root.*

Thus, the *principal cube root* of -27 is $(-27)^{\frac{1}{3}}$ or $\sqrt[3]{-27} = -3$.

III. If a is a *negative* number, and n is an *even* integer, then all the roots are imaginary. Imaginary numbers will be discussed more fully later.

Thus, $\sqrt[6]{-64}$ or $(-64)^{\frac{1}{6}}$ represents imaginary roots.

POWERS AND ROOTS

39. It is readily seen that

$$(a^3)^4 = a^3 \cdot a^3 \cdot a^3 \cdot a^3 = a.^{12}$$

This result may be obtained at once by multiplying the exponents 3 and 4. In this process algebraic expressions are raised to powers; it is called *involution.*

In general,

$$(a^m)^n = a^{mn}$$

$$(a^m b^n)^p = a^{mp} b^{np}$$

$$((a^m)^n)^p = a^{mnp}.$$

It is agreed that each of the symbols $a^{\frac{1}{n}}$, $\sqrt[n]{a}$, $b^{\frac{1}{n}}$, $\sqrt[n]{b}$, $(ab)^{\frac{1}{n}}$, $\sqrt[n]{ab}$ shall represent only one nth root, namely, the *principal*

nth root. For the principal real roots the following formulas can be shown to be true:

$$(a^{mn})^{\frac{1}{n}} = a^m \text{ or } \sqrt[n]{a^{mn}} = a^m.$$

$$a^{\frac{1}{n}} \cdot b^{\frac{1}{n}} = (ab)^{\frac{1}{n}} \text{ or } \sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}.$$

$$\frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}} = \left(\frac{a}{b}\right)^{\frac{1}{n}} \text{ or } \frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}.$$

Express each of these formulas in words.

The square root of 2 cannot be exactly expressed by the Hindu-Arabic numerals. One can *approximate* its value by extracting the square root to two, three, or more decimal places, thus: $\sqrt{2} = 1.41^+$, $\sqrt{2} = 1.414^+$, $\sqrt{2} = 1.4147^+$, and so on. The radical $\sqrt{2}$ or $2^{\frac{1}{2}}$, and other radicals of the same kind, $\sqrt{3}$ or $3^{\frac{1}{2}}$, $\sqrt[3]{5}$ or $5^{\frac{1}{3}}$, etc., whose values can be found approximately, but not exactly, represent numbers called *irrational* numbers.

ORAL EXERCISES

40. Express with fractional exponents and simplify:

1. $\sqrt{3\,mn^2}$.
2. $\sqrt[3]{2\,a^2b^6}$.
3. $\sqrt[3]{-27\,a^9c^2}$.
4. $a\sqrt{a^2x^3}$.
5. $2\sqrt{2\,x^5y^6}$.
6. $9\sqrt[4]{16\,c^2d^8}$.
7. $3\sqrt{\dfrac{a^3}{b^7}}$.
8. $-7\sqrt{\dfrac{x^3}{y}}$.
9. $\sqrt[3]{5 \cdot \sqrt[n]{a^m}}$.
10. $\sqrt[4]{5} \cdot \sqrt[4]{35}$.
11. $\sqrt[3]{a} \cdot \sqrt[6]{a^4}$.
12. $6\sqrt[7]{2\,a^5b^3c^2}$.

Express with radical signs:

13. $2(7\,y)^{\frac{2}{3}}$.
14. $5\,x^{\frac{1}{2}}y^{\frac{3}{2}}$.
15. $3^{\frac{1}{3}}x^{\frac{4}{3}}y^{\frac{3}{2}}$.
16. $\dfrac{5^{\frac{1}{2}}}{m^{\frac{3}{4}}}$.
17. $5(x+y)^{\frac{4}{3}}$.
18. $4(x^2-y^2)^{\frac{3}{2}}$.

Simplify:

19. $49^{-\frac{1}{2}}$.

20. $81^{\frac{1}{2}}$.

21. $5^0 \cdot 16^{\frac{1}{4}}$.

22. $16^{-\frac{1}{4}} \cdot 25^{\frac{1}{2}}$.

23. $(\frac{1}{25})^{-\frac{1}{2}}$.

24. $(\frac{1}{64})^{\frac{1}{3}}$.

25. $\dfrac{25^{\frac{1}{2}}}{8^{\frac{2}{3}}}$.

26. $13^2 \cdot 26^{-2}$.

27. $\dfrac{a^{-1}b^2}{4^{-2}a^{-1}}$.

28. $a^3(a^{-2} + c^{-2})$.

29. $a^{-3}(a^4 + b)$.

30. $36^{-\frac{1}{2}} \cdot 343^{\frac{1}{3}}$.

31. $(-3)^{-2} \cdot (-3)^5$.

32. $(\frac{25}{49})^{-\frac{1}{2}}$.

33. $(\frac{25}{49})^{-\frac{3}{2}}$.

34. $a^0 \cdot (\frac{9}{4})^{-\frac{1}{2}}$.

35. $\dfrac{1}{27^0} \cdot 27^{\frac{4}{3}}$.

36. $\dfrac{a^{-3}b^{-4}}{a^2b^{-5}}$.

37. $\dfrac{a^{-4}(b + c)}{c - 1}$.

38. $\dfrac{a^{-3} + b}{a + b^{-3}}$.

Multiply numerator and denominator by a^3b^3.

39. $\dfrac{m^{-2} + b^{-2}}{m^{-2} - b^{-2}}$.

Find the principal roots of:

40. $\sqrt{x^4y^6}$.

41. $\sqrt{4\,a^2b^6}$.

42. $\sqrt{9\,x^4y^8}$.

43. $\sqrt{25\,x^2y^{10}}$.

44. $\sqrt{100\,a^{16}}$.

45. $\sqrt[3]{-8\,a^3b^{12}}$.

46. $\sqrt[3]{64\,x^6y^3z^{15}}$.

47. $\sqrt[3]{-343\,a^3b^{18}}$.

48. $\sqrt[3]{\dfrac{8\,x^9}{729\,y^3}}$.

49. $\sqrt{\dfrac{256\,a^2b^4}{289\,m^{14}}}$.

50. $\sqrt[8]{256\,x^8y^{56}}$.

51. $\sqrt{\dfrac{81\,a^{28}b^{36}}{400\,x^{10}y^{16}}}$.

52. $\sqrt[5]{-x^{10}y^{20}}$.

53. $\sqrt[7]{\dfrac{128\,a^7}{x^{14}y^{21}}}$.

54. $\sqrt[6]{729\,x^{12}y^{18}z^{24}}$.

55. $\sqrt[9]{\dfrac{-r^2s^{18}}{t^{45}u^{27}}}$.

56. $\sqrt[3]{125\,x^{15}y^3}$.

57. $\sqrt[10]{\dfrac{a^{20}b^{30}}{c^{50}}}$.

58. $\sqrt[3]{\dfrac{-512\,a^{21}}{27\,x^{27}}}$.

59. $\sqrt[n]{\dfrac{a^nb^n}{c^{2n}d^{3n}}}$.

60. $\sqrt[r]{\dfrac{x^{2r}y^{4r}}{7^{5r}}}$.

Find the *two real* fourth roots of:

61. $10{,}000\,a^2b^3c^4(a + b)^4$.

62. $625\,m^8n^6p^3(a^2 - b^2)^2$.

63. $\frac{1}{16}\,x^{-4}y^{-12}z^8(x - y)^4$.

64. $.0081(x - y)^8z^4w^{-12}$.

Find *the real* fifth root of:

65. $-32\,x^5(m+n-p)^{-5}y^3z^4$. **66.** $243\,a^{10}(b-c)^{15}x^{-5}z^4$.

67. What kind of roots of positive numbers have two real values? Which of those two real roots is the principal root?

68. What kind of roots of positive or negative numbers have only one real value?

69. What kind of roots of negative numbers have no real value?

SQUARE ROOT

41. Since
$$(a+b)^2 = a^2 + 2\,ab + b^2,$$
$$\sqrt{a^2 + 2\,ab + b^2} = a + b.$$

Thus, by inspection, we can extract the square root of a trinomial which is a perfect square.

Or we may find the square root in this way:

$$\begin{array}{lr|l} & a^2 + 2\,ab + b^2 & \underline{a+b} \\ & \underline{a^2} & \\ \text{Trial divisor,} & 2\,a & 2\,ab + b^2 \\ \text{Complete divisor,} & 2\,a + b & \underline{2\,ab + b^2} \end{array}$$

Thus the square root is $a + b$.

This simple case will enable us to devise a rule which is applicable to more complicated cases. The procedure is as follows:

I. Extract the square root of the first term and subtract its square from the polynomial, leaving $2\,ab + b^2$.

II. In $2\,ab$, we see the factor b which we know is the second term of the root. This factor b may be obtained by dividing $2\,ab$ by $2\,a$. Thus we call $2\,a$ the trial divisor. Since a is the part of the root already found, we see that the *trial divisor is double the root already found.* After b has been found, add it to the trial divisor and we have $2\,ab + b$, the complete divisor.

III. Multiply $2\,ab + b$ by b, and subtract.

This process may be extended to finding the square root of any polynomial.

Find the square root of $x^4 + 21\,x^2 - 36\,x + 36 - 6\,x^3$. Arrange the terms according to the ascending or descending powers of some letter,

$$
\begin{array}{lll}
 & x^4 - 6\,x^3 + 21\,x^2 - 36\,x + 36 & \underline{|\,x^2 - 3\,x + 6} \\
 & \underline{x^4} & \\
\text{1st trial divisor,} \quad 2(x^2) = 2\,x^2 & |\,-6\,x^3 + 21\,x^2 - 36\,x + 36, & \text{1st remainder} \\
\text{1st complete divisor,} \quad 2\,x^2 - 3\,x & |\,-6\,x^3 + \ \ 9\,x^2 & \\
\text{2d trial divisor,} \quad 2(x^2 - 3\,x) = 2\,x^2 - 6\,x & |\,12\,x^2 - 36\,x + 36, & \text{2d remainder} \\
\text{2d complete divisor,} \quad 2\,x^2 - 6\,x + 6 & |\,\underline{12\,x^2 - 36\,x + 36} & \\
 & \qquad\qquad 0 & \text{3d remainder}
\end{array}
$$

The procedure is as follows:

I. The square root of x^4 is x^2. Here $a = x^2$; $2\,a = 2\,x^2$, the first trial divisor.

II. $2\,ab = -6\,x^3$; divide $-6\,x^3$ by $2\,x^2$ and we obtain $-3\,x$, the value of b. Thus the first complete divisor is $2\,x^2 - 3\,x$.

III. Multiply $2\,x^2 - 3\,x$ by $-3\,x$ and we obtain $-6\,x^3 + 9\,x^2$.

Subtract this product from the 1st remainder and we obtain

$$12\,x^2 - 36\,x + 36.$$

IV. Treat this remainder as we did the first remainder and proceed as before.

Since the third remainder is 0, $x^2 - 3\,x + 6$ is the required square root.

EXERCISES

42. Find the square root of:

1. $4\,a^6 - 28\,a^3b^6 + 49\,b^{12}$.
2. $4\,a^2 - 12\,ab + 4\,ac + 9\,b^2 - 6\,bc + c^2$.
3. $x^2 - 2\,xy + 6\,x + y^2 - 6\,y + 9$.
4. $4\,x^4 - 20\,x^3 - 30\,x + 9 + 37\,x^2$.
5. $9\,x^6 - 2\,x^4 - 3\,x^2 + 1 + 10\,x^3 - 12\,x^5 - 2\,x$.
6. $9\,a^6 + 9\,b^6 + 24\,a^5b + 24\,ab^5 - 8\,a^4b^2 - 8\,a^2b^4 - 50\,a^3b^3$.
7. $\dfrac{x^2}{4} - 5\,x + 25$.
8. $\dfrac{a^2}{9} - 2 + \dfrac{9}{a^2}$.

9. $\frac{4}{9}m^4 - m^3 + \frac{187}{48}m^2 - \frac{15}{4}m + \frac{25}{4}$.

10. $4 - \frac{4a}{b} + \frac{a^2}{b^2} + \frac{2a^3}{3b} - \frac{4a^2}{3} + \frac{a^4}{9}$.

11. $a^2 + ab$, to three terms.

12. $1 - a$, to four terms.

SQUARE ROOT OF ARITHMETICAL NUMBERS

43. $1^2 = 1$, $2^2 = 4$, $\cdots$, $9^2 = 81$, $10^2 = 100$, $\cdots$, $90^2 = 8100$, $100^2 = 10000$, $1000^2 = 1{,}000{,}000$.

This is sufficient to show that a perfect square consisting of *one* or *two* digits has *one* digit in the root; one consisting of *three* or *four* digits has *two* digits in the root; one consisting of *five* or *six* digits has *three* digits in the root; and so on. In other words, the square of a number has twice as many integral digits, or one less than twice as many, as the number itself.

Hence, to find the square root we must separate the number into *periods* of *two* digits each, *towards the left and right from the decimal point.* The period farthest to the left may have one or two digits, whereas the one farthest to the right must have two, a cipher being annexed, if necessary, to complete it.

The square root has therefore one digit corresponding to every period in its square. Separating the square number into periods enables one to find one digit in the root at a time.

This is seen more clearly if we consider that the square of any number of tens, say 4 tens (40), ends in two ciphers (1600); hence the two digits on the right are not needed to find the tens' digit (4), and are set aside until the unit's digit is to be found.

Likewise, the square of any number of hundreds (400), ends in four ciphers (160,000), and all of these may be set aside until the hundreds' digit is found and they are needed for the tens' and units' digits.

After pointing off the number into periods, the method is similar to the one used for polynomials. The procedure is based on the formula $a^2 + 2\,ab + b^2 = (a + b)^2$.

When dividing by the trial divisor, *exclude, for brevity, the right-hand digit in the remainder.*

The reason for this is seen in the example which follows: the trial divisor 4 is equal to 40 of the next lower units. Now 14 divided by 4 gives the same digit as $148 \div 40$.

Point off one decimal place in the root for each period in the decimal.

Find the square root of 548.543241.

		$5\dot{4}8.5\dot{4}3\dot{2}4\dot{1}$	23.421
		4	
1st trial divisor, 2 (2) = 4		148	1st remainder.
1st complete divisor,	43	129	
2d trial divisor, 2(23) = 46		1954	2d remainder.
2d complete divisor,	464	1856	
3d trial divisor, 2(234) =	468	9832	3d remainder.
3d complete divisor,	4682	9364	
4th trial divisor, 2(2342) =	4684	46841	4th remainder.
4th complete divisor,	46841	46841	
		0	5th remainder.

Point off three decimal places in the root.

Thus, the square root is 23.421.

To find the square root of a fraction, which is not a perfect square: Reduce the fraction to a decimal and find the square root of the decimal.

EXERCISES

44. Find the square root of:

1. 6889.
2. 2025.
3. 161,604.
4. 337,561.
5. 7.3984.
6. 567,009.
7. 2777.29.
8. 88.454025.
9. .00698896.
10. .059049.

Find the square root, to 3 decimal places, of:

11. 2. 13. 5. 15. $3\frac{2}{3}$.

12. 3. 14. 1.005. 16. $\frac{7}{8}$.

FACTORING

45. *Factoring* is the process of finding two or more expressions whose product is a given expression.

An *integral expression* is one that contains no fractions.

A *rational integral algebraic expression* is one containing no fractions, no negative exponents, and no indicated roots.

$2 + 3ax + x^3$ is a rational integral expression; $\frac{1}{3}x^2 + \frac{2}{5}x + 1$, $x^{-2} + 5$, $\sqrt{a+x} + y$, $x^{\frac{1}{3}} + 10$, are not rational integral expressions.

A *prime factor* of an integer is an integral factor which is exactly divisible only by *itself* or *one*.

Thus, 5 and 13 are prime factors of 65.

A *prime factor* of a rational integral expression is a rational integral factor which cannot itself be resolved into rational integral factors other than *itself* and *one*.

Thus, $a - b$ and $a^2 + ab + b^2$ are prime factors of $a^3 - b^3$.

In the following exercises we shall confine ourselves to finding *prime* factors of rational integral expressions. In § 83 irrational factors will be required.

I. Type form: $\boldsymbol{ab + ac - ad}$.

$$ab + ac - ad = a(b + c - d).$$

46. Factor:

1. $y^4 - 2y^3 + 3y^2 - y$.
2. $80x^3y^4 - 160x^2y^5 - 240xy^6$.
3. $a(x + y) - b(x + y)$.
4. $(x - y)c - (x + y)c$.
5. $m(c - d) - n(c - d) + p(c - d)$.
6. $a(m - n) + b(m - n)$.
7. $m(a - b) - n(b - a)$.

8. $a(x-3y)+b(3y-x)$.
9. $(x-y)-2a(y-x)+3b(y-x)$.
10. $c(a-b-c)+d(b+c-a)$.

II. Type form: $\boldsymbol{ax+bx+ay+by}$.

$$ax+bx+ay+by=a(x+y)+b(x+y)=(x+y)(a+b).$$

47. Factor:

1. $3x+3y+ax+ay$.
2. $ax-ay-bx+by$.
3. $2ax-2bx+b-a$.
4. $4a-4b+ac-bc$.
5. $2ac-3bc-2ad+3bd$.
6. $a^nc^n-b^nc^n+a^nd^n-b^nd^n$.
7. $a^2-ab+ac-ac+bc-c^2$.
8. $a^4x^4+b^4x^4-a^4y^4-b^4y^4$.

III. Type form:

$$\boldsymbol{a^2\pm 2ab+b^2} \text{ and } \boldsymbol{a^2+b^2+c^2+2ab-2ac-2bc}.$$

$$a^2\pm 2ab+b^2=(a\pm b)^2.$$
$$a^2+b^2+c^2+2ab-2ac-2bc=(a+b-c)^2.$$

48. Factor:

1. $9r^2-24rs+16s^2$.
2. $a^{2n}-14a^n+49$.
3. $a^2+a+\frac{1}{4}$.
4. $(x-y)^2-2(x-y)+1$.
5. $4(c-3)^2-12(c^2-9)+9(c+3)^2$.
6. $x^2+y^2+z^2-2xy-2xz+2yz$.
7. $a^2+b^2+c^2-2ab+2ac-2bc$.
8. $1+a^2+b^2+2a-2b-2ab$.
9. $4a^2+b^2+c^2-4ab+4ac-2bc$.
10. $m^2+n^2+9-2mn-6m+6n$.

IV. Type form: $\boldsymbol{a^2-b^2}$.

49.
$$a^2-b^2=(a+b)(a-b).$$
$$a^2-2ab+b^2-c^2=(a-b)^2-c^2=(a-b+c)(a-b-c).$$
$$a^2-c^2-2cd-d^2=a^2-(c+d)^2=(a+c+d)(a-c-d).$$
$$a^2-2ab+b^2-c^2+2cd-d^2=(a-b)^2-(c-d)^2$$
$$=(a-b+c-d)(a-b-c+d).$$

50. Factor:

1. $c^3 - 64$.
2. $121\,c^2 - 144\,d^4$.
3. $x^2 - m^2 - 2\,mn - n^2$.
4. $49\,a^6 - 64 + 16\,b - b^2$.
5. $100\,a^{10} - a^2 + 12\,ab - 36\,b^2$.
6. $36\,y^2 + 84\,y + 49 - 9\,z^2$.
7. $m^6 + 10\,m^3 + 25 - 81\,n^2$.
8. $(a-3)^2 - (b-2)^2$.
9. $4\,b^2 - x^2 + 4\,xy + a^2 + 4\,ab - 4\,y^2$.
10. $25\,d^2 - 1 - 9\,c^2d^2 - 10\,ad + a^2 + 6\,cd$.

V. Type form: $\boldsymbol{x^2 + bx + c}$.

$$x^2 + bx + c = (x + r)(x + s)\,;\ rs = c,\ r + s = b.$$

51. Factor:

1. $x^2 + 5\,x + 6$.
2. $x^2 - 5\,x + 6$.
3. $y^2 - 7\,y + 6$.
4. $y^2 + 7\,y - 98$.
5. $z^2 + 10\,z + 9$.
6. $r^2 - 4\,rs - 77\,s^2$.
7. $b^2c^2 + 7\,bcy + 10\,y^2$.
8. $c^2d^2 + 7\,cd^2 - 120\,d^2$.
9. $t^6 - 23\,t^2 - 50$.
10. $x^6y^8 - 16\,x^3y^4z + 39\,z^2$.

VI. Type form: $\boldsymbol{ax^2 + bx + c}$.

52. $ax^2 + bx + c = (px + q)(rx + s)$, wherein $pr = a$, $ps + qr = b$, $qs = c$.

By this method p, q, r, s must be found by trial, so that the three conditions just named will be satisfied.

Factor $6\,x^2 - 5\,x - 6$.

Here $a = 6$, $b = -5$, $c = -6$.

If possible, we must find numbers p, q, r, s, such that $pr = 6$, $qs = -6$, $ps + qr = -5$.

$pr = 6$ suggests the values $p = 3$, $r = 2$, or $p = 1$, $r = 6$, etc.

$qs = -6$ suggests the values $q = 3$, $s = -2$, or $q = 2$, $s = -3$, etc.

Try the different sets of values of p and r, whose product is 6, with each pair of values of q and s, whose product is -6, until a combination is found which satisfies $ps + qr = -5$.

Try the sets of values $p = 3$, $r = 2$, $q = 2$, $s = -3$.

Examine
$$\overbrace{(3x+2)(2x-3)}^{+4x}$$
$$\underbrace{\qquad\qquad}_{-9x}$$

We see that the sum of the two cross products $+4x$ and $-9x$ is $-5x$, which is the middle term of $6x^2 - 5x - 6$. Hence

$$6x^2 - 5x - 6 = (3x+2)(2x-3).$$

53. Factor:

1. $6x^2 + 5x - 6$.
2. $5x^2 + 14x - 3$.
3. $5x^2 - 14x - 3$.
4. $8y^2 - 14y - 39$.
5. $18m^2 + 37m + 19$.
6. $15x^2 + 14ax - 8a^2$.
7. $17a^2x^2 - 36ax + 4$.
8. $20r^2 + 19rs + 3s^2$.
9. $9x^2 - 36xy - 13y^2$.
10. $28m^2 - 48m + 17$.

VII. Type form: $a^4 + a^2b^2 + b^4$.

$$a^4 + a^2b^2 + b^4 = a^4 + 2a^2b^2 + b^4 - a^2b^2 = (a^2+b^2)^2 - a^2b^2$$
$$= (a^2 + ab + b^2)(a^2 - ab + b^2).$$

54. Factor:

1. $x^4 + x^2 + 1$.
2. $9m^4 + 8m^2n^2 + 4n^4$.
3. $16a^4 + 20a^2b^2 + 9b^4$.
4. $r^8 + r^4s^4 + s^8$.
5. $a^4 - 5a^2b^2 + 4b^4$.
6. $x^4 - 14x^2y^2 + y^4$.
7. $m^8 - 38m^4n^4 + n^8$.
8. $4a^4 - 13a^2b^2 + 9b^4$.
9. $25c^4 + 26c^2d^2 + 9d^4$.
10. $49a^4 + 110a^2b^2 + 81b^4$.

55. In factoring, first take out monomial factors. Then inspect the resulting polynomial, ascertaining to which type form it belongs, and factor it accordingly. In the final form, all factors should be *prime*.

ORAL EXERCISES

56. Factor:

1. $m^2 - a^2$.
2. $4x^2 - y^2$.
3. $a^2 + 6ab + 9b^2$.
4. $9x^2 - 6xy + y^2$.
5. $4x^2 - 12xz + 9z^2$.
6. $ax^2 - 16ay^2$.
7. $cx^2 + cy^2$.
8. $y^2 - 5y + 6$.
9. $20 - 9z + z^2$.
10. $m^4 - n^4$.
11. $pq^2 - p^2q$.
12. $x^2 + 2x - 35$.
13. $x^2 - 2x - 35$.
14. $m^2 - 2m - 3$.
15. $p^2 - 6p + 5$.
16. $4c^2 \pm 4cd + d^2$.
17. $x^3 + y^2x$.
18. $x^3 - y^2x$.
19. $x^3 - 4xy^2$.
20. $4a^3 - ab^2$.
21. $a^3b - 9b^3$.
22. $5a^5 + a^4b$.
23. $a^4 - 2a^2b^2 + b^4$.
24. $9a^2 \pm 12ab + 4b^2$.

WRITTEN EXERCISES

57. Factor:

1. $4a^2 - 20ab$.
2. $b^2 - c^2$.
3. $m^4 - 9n^4$.
4. $4a^4 - 8a^2b$.
5. $a^3bc - ab^3c$.
6. $25r^4 - 36s^4$.
7. $y^3 + z^3 - y - z$.
8. $m^2n^4 - 2mn^2 + 1$.
9. $16 + 8c + c^2$.
10. $1 - a^6$.
11. $a^{12} - b^4$.
12. $a^2 - 5ab + 6b^2$.
13. $cx + cy - dx - dy$.
14. $b^4 + 6b^2 + 5$.
15. $64 - b^8$.
16. $7x - 12 - x^2$.
17. $a^{10} - a^2$.
18. $3x^3 + 3x^2 - 27x - 27$.
19. $x^{2a} + 2x^a - 15$.
20. $x^3 - 6x^2 + 12x - 8$.
21. $a^2 - a + \frac{1}{4}$.
22. $r^2 - s^2 - 2st - t^2$.
23. $25 - 4k^4 + 4h^2k^2 - h^4$.
24. $m^2 - 2mn + n^2 + m - n$.
25. $r^4 + r^3 + r^2 + r$.
26. $x^4 + 2x^2 + 1$.
27. $c^2 - d^2 - c - d$.
28. $b^2 - 2bc + c^2 + 5b - 5c$.
29. $6a^2 + 13ab + 6b^2$.
30. $x^{13} - xy^{12}$.
31. $r^4 - 4r^3 + 6r^2 - 4r + 1$.
32. $(x - y)^2 - 9(x + y)^2$.
33. $a^2 + 12ab + 36b^2$.
34. $(a + b)^2 - 7(a + b) + 12$.
35. $(x^2 - 13x + 42)(x^2 + 3x - 10)$.
36. $(b - c)^2 + 36 - 12(b - c)$.

37. $(a+d)^2+(b+c)^2-2(a+d)(b+c)$.

38. $25\,a^4-41\,a^2b^2+16\,b^4$.

39. $a^2-2\,ab+b^2-2\,ac+2\,bc+c^2$.

40. $c^{16}-d^{16}$.

41. $x^2-49\,x+360$.

42. $r^4-47\,r^2+1$.

43. $ab-abc^2d^2$.

44. $m^4+64\,n^4$.

45. $(m-n)^2-1+a(m-n+1)$.

46. $3\,a^2-10\,ab-8\,b^2+3\,ad+2\,bd-9\,ac-6\,bc$.

47. $x^4-3\,x^2+4$.

48. $4\,a^4+81\,b^4$.

49. $4\,a^4-37\,a^2b^2+9\,b^4$.

50. $64\,y^4+128\,y^2z^2+81\,z^4$.

FRACTIONS

58. A *fraction* is the indicated quotient obtained by dividing one number by another.

$$\frac{x}{y},\ \frac{2\,x-y}{z},\ \frac{x^2-y^2}{x^2+y^2} \text{ are fractions.}$$

The fundamental principle of operations with fractions is — *Both numerator and denominator of a fraction may be multiplied or divided by the same number, without changing the value of the fraction.*

Thus, $$\frac{7}{8}=\frac{14}{16},\ \frac{2\,x}{2\,a}=\frac{x}{a},\ \frac{8(x+y)}{4(x+y)(x-y)}=\frac{2}{x-y}.$$

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EXERCISES

59. Reduce to lowest terms:

1. $\dfrac{3a^2bc^3}{6a^3bc^2}$.

2. $\dfrac{64x^5y^9}{16x^7y^3}$.

3. $\dfrac{2a-2b}{a^2-b^2}$.

4. $\dfrac{5ax-10ay}{x^2-5xy+6y^2}$.

5. $\dfrac{8(x^2-b^2)}{24x+24b}$.

6. $\dfrac{x^2-4x+4}{3x^2+3x-18}$.

7. $\dfrac{(a-b)^2(a+b)^2}{(a^2-b^2)^2}$.

8. $\dfrac{4-(m-n)^2}{n^2-(m-2)^2}$.

9. $\dfrac{a^3-2a-1}{a^3-2a^2+1}$.

10. $\dfrac{a^2-c^2-9-2ab-6c+b^2}{b^2-a^2-9-2bc-6a+c^2}$.

60. From the laws of signs in multiplication and division it is evident that

$$\frac{a}{b}=\frac{-a}{-b},\quad \frac{a}{b}=-\frac{-a}{b},\quad \frac{a}{b}=-\frac{a}{-b}.$$

From these relations we see that

1. The signs in *both numerator and denominator* may be changed without changing the **value** of the fraction.

2. The sign *of the numerator and of the fraction* may be changed without changing the **value** of the fraction.

3. The sign of the *denominator and of the fraction* may be changed without changing the **value** of the fraction.

61. The following principles may be used effectively in operations with fractions.

1. The signs of an *even* number of *factors* may be changed *without* changing the sign of the product. Explain.

Thus, $a\cdot b\cdot c=(-a)\cdot(-b)\cdot c=abc.$

2. The signs of an *odd* number of *factors* may be changed, provided the sign of the product is changed. Explain.

Thus, $a\cdot b\cdot c=-(-a)\cdot b\cdot c=abc$; $a\cdot b\cdot c\neq(-a)(-b)(-c)$.

MULTIPLICATION AND DIVISION OF FRACTIONS

62. In multiplication and division of fractions in algebra we have the same rules of operation as in arithmetic; namely,

1. *To find the product of two fractions, multiply the numerators together for a new numerator, and the denominators together for a new denominator.*

2. *To find the quotient, invert the divisor and then proceed as in the multiplication of a fraction by a fraction.*

It is easy to establish the truth of these rules by means of the equation.

For example, the second rule may be proved as follows:

Let $$\frac{a}{b} = x, \quad \frac{c}{d} = y;$$

then $$a = bx, \quad c = dy,$$

and $$ad = bdx, \quad bc = bdy.$$

Dividing equals by equals,

$$\frac{ad}{bc} = \frac{bdx}{bdy} = \frac{x}{y}.$$

But $$\frac{x}{y} = \frac{a}{b} \div \frac{c}{d}.$$

Hence $$\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}.$$

This proves the rule for the division of one fraction by another.

Proceeding in a similar manner, prove the rule for the multiplication of fractions.

Multiply $$\frac{axy + ay^2}{bx + by} \text{ by } \frac{bx^2 - by^2}{ax + ay}.$$

We have $$\frac{\cancel{d}y\cancel{(x+y)}}{\cancel{b}\cancel{(x+y)}} \cdot \frac{\cancel{b}\cancel{(x+y)}(x-y)}{\cancel{d}\cancel{(x+y)}} = y(x-y).$$

EXERCISES

63. Perform the indicated operations:

1. $\frac{a}{b} \cdot \frac{b}{c} \div \frac{d}{c}$.

2. $\frac{2a}{x} \cdot \frac{3bx}{a} \div \frac{2b}{3ax}$.

3. $\frac{5a}{2b-2} \div \frac{2a}{b-1}$.

4. $\frac{16m^4n}{45xy^3} \div \frac{8m^3}{9xy^2}$.

5. $\frac{x^2+y^2}{x^2-y^2} \div \frac{x-y}{x+y}$.

6. $\frac{r-s}{r^2+rs} \cdot \frac{r^2-s^2}{r^2-rs}$.

7. $\frac{x^2-3x+2}{x^2-6x} \div \frac{x^2-3x}{x^2+x-42}$.

8. $\frac{a-b}{a+b} \div \frac{(a-b)^2}{(a+b)^2}$.

9. $\frac{a^2+ab}{a^4-y^4} \div \frac{a+b}{(a-y)^2}$.

10. $\frac{c^2-d^2}{c^2+d^2} \cdot \frac{c+d}{d-c}$.

11. $\frac{x^2-4x+3}{x^2-7x+10} \cdot \frac{x^2-12x+35}{x^2+2x-3} \cdot \frac{5x-10}{x^2-3x}$.

12. $\frac{a^2+ab}{a^2+b^2} \div \frac{a^3c+2a^2bc+ab^2c}{a^2c-b^2c}$.

13. $\frac{m^4-n^4}{3m^2+3n^2} \cdot \frac{6(m^3-m^2n+mn^2-n^3)}{m^2-2mn+n^2}$.

14. $\frac{18a^2-9a}{7a^3+21a^2} \div \frac{2a^2+5a-3}{(a+3)^2}$.

15. $\frac{r^2+3r-10}{r^2-1} \div \frac{r^2-12r+20}{r^2-11r+10}$.

16. $\frac{a^2-c^2+2ab+b^2}{a^2-c^2-2bc-b^2} \cdot \frac{a-b-c}{a+b-c}$.

17. $\frac{2a^2+a-10}{a^2-4} \cdot \frac{3a^2+4a-4}{2-3a}$.

18. $\dfrac{x^4 - y^4}{(x+y)^2} \cdot \dfrac{x^2 - xy}{x+y} \div \dfrac{x^2 - 2\,xy + y^2}{x^2 + 2\,xy + y^2}.$

19. $\dfrac{9\,a^2 - 16\,b^2}{3\,a^2 - 11\,ab - 20\,b^2} \div \dfrac{6\,a^2 - 8\,ab}{a^2 - 5\,ab}.$

20. $\dfrac{b^2 - 13\,b + 42}{ab - 3\,b - 7\,a + 21} \cdot \dfrac{a^2 + 6\,a - 27}{a^2 - 81}.$

COMPLEX FRACTIONS

64. *A complex fraction* is one which contains one or more fractions in its numerator, or denominator, or both.

Complex fractions may be simplified in two ways:

1. *Simplify the numerator and denominator, then divide the numerator by the denominator.*

2. *Multiply both numerator and denominator by the l. c. d. of their fractional terms.*

The pupil is probably familiar with the first method from his study of arithmetic. In simplifying complex fractions arising in algebra, the second method is usually the easier. It should be thoroughly mastered.

Simplify, $\dfrac{\frac{1}{2} - \frac{1}{3}}{\frac{1}{4} - \frac{1}{9}}.$

The l. c. m. of 2, 3, 4, 9 is 36.

Multiplying the numerator of the complex fraction by 36 gives 18 − 12.

Multiplying the denominator of the complex fraction by 36 gives 9 − 4.

Hence, $\dfrac{\frac{1}{2} - \frac{1}{3}}{\frac{1}{4} - \frac{1}{9}} = \dfrac{18 - 12}{9 - 4} = \dfrac{6}{5}.$

EXERCISES

65. Perform the indicated operations and simplify:

1. $\left(\dfrac{a}{b} - \dfrac{b}{a}\right) \div \left(a - \dfrac{b^2}{a}\right).$

2. $\left[\dfrac{a(x-y)}{a+b} + y\right] \div \dfrac{by + ax}{a^2 - b^2}.$

3. $\dfrac{3}{1-a^2} \div \left(\dfrac{1}{1-a} - \dfrac{1}{1+a}\right).$

4. $(2x^2 - 5x + 3) \div \left(\dfrac{1}{x^2} - 1\right).$

5. $\left(\dfrac{1}{xy} + \dfrac{1}{yz} + \dfrac{1}{xz}\right)\left(\dfrac{x}{y} + \dfrac{y}{z} + \dfrac{z}{x}\right) \div \dfrac{x+y+z}{x^3y^3z^3}.$

6. $\left(\dfrac{1}{a} - a^2\right) \div \{(a+1)^2 - a\}\left\{a + \dfrac{a^3}{1-a^2}\right\}.$

7. $\left(\dfrac{m}{n} + \dfrac{r}{s}\right) \div \left(\dfrac{n}{m} + \dfrac{s}{r}\right).$

8. $\left(\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z}\right) \div \left(\dfrac{x}{y} + \dfrac{y}{z} + \dfrac{z}{x}\right).$

9. $\left(\dfrac{a}{b^2} - \dfrac{b}{a^2}\right) \div \left(\dfrac{1}{a^2} + \dfrac{1}{ab} + \dfrac{1}{b^2}\right).$

10. $(m^2 - 6m - 7) \div \dfrac{m-7}{m+1} \cdot \left(\dfrac{m^2}{(m+1)^2} - \dfrac{m-1}{m+1}\right).$

11. $\dfrac{2}{1+\dfrac{1}{x}}.$

12. $\dfrac{\dfrac{a}{b}}{a+\dfrac{a}{b}}.$

13. $\dfrac{\dfrac{1}{a} - \dfrac{1}{b}}{\dfrac{b}{a} + \dfrac{a}{b}}.$

14. $\dfrac{2+\dfrac{2}{a}}{2-\dfrac{2}{b}}.$

15. $\dfrac{1+\dfrac{r}{s}}{1-\dfrac{r^2}{s^2}}.$

16. $\dfrac{\dfrac{a+b}{a-b} - \dfrac{a-b}{a+b}}{\dfrac{a-b}{a+b} + \dfrac{a+b}{a-b}}.$

17. $$\frac{x-3-\dfrac{4}{x+4}}{x-5-\dfrac{3}{x+4}}.$$

18. $$\frac{x+y+\dfrac{y^2}{x}}{x+y+\dfrac{x^2}{y}}.$$

19. $$\frac{\dfrac{a^2+ab}{a^2-b^2}}{\dfrac{a}{a+b}+\dfrac{b}{a-b}}.$$

20. $$\frac{\dfrac{r}{3+r}-\dfrac{3-r}{r}}{\dfrac{r}{r+3}+\dfrac{3-r}{r}}.$$

HIGHEST COMMON FACTOR AND LEAST COMMON MULTIPLE

66. Only expressions which contain no fractions or radicals will be considered here.

The *highest common factor* (h. c. f.) of two or more expressions is the product of the prime factors that occur in *each* expression, every factor being taken the least number of times it occurs in any one expression.

This definition indicates the process of finding the h. c. f.

The *lowest common multiple* (l. c. m.) of two or more expressions is the product of all the prime factors that occur in the expressions, every factor being taken the *greatest* number of times it occurs in any one expression.

This definition indicates the process of finding the l. c. m. Observe that the l. c. m. is the least expression which is exactly divisible by each of the given expressions, while the h. c. f. is the expression of highest degree by which each of the given expressions is exactly divisible.

EXERCISES

67. 1. Find the h. c. f. and l. c. m. of

$$4\,ax-4\,ay,\ 8\,a^2x^2-8\,a^2y^2,\ 12\,x^3-12\,y^3.$$

Factoring, $4\,ax-4\,ay=2^2a(x-y).$

$8\,a^2x^2-8\,a^2y^2=2^3a^2(x+y)(x-y).$

$12\,x^3-12\,y^3=2^2\cdot 3(x-y)(x^2+xy+y^2).$

The h. c. f. $=2^2(x-y)=4(x-y).$

The l. c. m. $=2^3\cdot 3\,a^2(x-y)(x+y)(x^2+xy+y^2)$

$=24\,a^2(x+y)(x^3-y^3).$

Find the h. c. f. and l. c. m. of:

2. $9\,abc$, $18\,a^2bc^2$, $27\,a^3b^2c$.

3. $15\,x^2y^2z^3$, $-30\,x^3y^3z$, $45\,x^3y^4z^2$.

4. $(a-b)^2$, a^2-b^2.

5. $3\,a^2+6\,ab+3\,b^2$, $9\,a^2-9\,b^2$.

6. $2(a-b)^2(a+b)$, $3(a+b)^3(a-b)$.

7. $x^2-5\,x+6$, $x^2+2\,x-8$.

8. $3\,a^3-3\,ab^2$, $3\,a(a+b)^2$.

9. $3\,ax^2-15\,ax+18\,a$, $6\,a^2x^2+24\,a^2x-126\,a^2$.

10. x^2-y^2, $x-y$, $ax-ay-cx+cy$.

11. $a(a-1)^2$, $a^2(a^2-1)$, $2\,a(a^2+2\,a-3)$.

12. $5(x^2-y^2)$, $10(x+y)^2$, $15(x-y)^2(x+y)$.

13. $h^2-5\,h-14$, $h^2-10\,h+21$, h^2-49.

14. $16(ab+b)$, $8\,a(a^2-b^2)$, $24\,ab(a^2+2\,ab+b^2)$.

15. $m^2(m-1)^2$, $m(m^2-1)$, $m^3(m^4n-n)$.

16. $(y+4)(y^2-16)$, y^2-y-20, $xy+4\,x-ay-4\,a$.

ADDITION AND SUBTRACTION OF FRACTIONS

68. The process is the same as in arithmetic. If the fractions do not have the same denominator, they must be reduced to fractions which do have the same denominator. The *lowest common denominator* (l. c. d.) is obtained by finding the *lowest common multiple* of the given denominators.

1. Perform the indicated addition and subtraction:

$$\frac{1-x}{2\,ax}+\frac{a+x}{3\,ax^2}-\frac{5\,a-x}{6\,a^2x}.$$

The l. c. d. is $6a^2x^2$. The reduction of the fractions to the l. c. d. is effected as follows:

$$6a^2x^2 \div 2ax = 3ax, \qquad \frac{1-x}{2ax} \cdot \frac{3ax}{3ax} = \frac{3ax - 3ax^2}{6a^2x^2}.$$

$$6a^2x^2 \div 3ax^2 = 2a, \qquad \frac{a+x}{3ax^2} \cdot \frac{2a}{2a} = \frac{2a^2 + 2ax}{6a^2x^2}.$$

$$6a^2x^2 \div 6a^2x = x, \qquad \frac{5a-x}{6a^2x} \cdot \frac{x}{x} = \frac{5ax - x^2}{6a^2x^2}.$$

We obtain $\dfrac{1-x}{2ax} + \dfrac{a+x}{3ax^2} - \dfrac{5a-x}{6a^2x} = \dfrac{3ax - 3ax^2 + 2a^2 + 2ax - 5ax + x^2}{6a^2x^2}$

$$= \frac{2a^2 - 3ax^2 + x^2}{6a^2x^2}, \textit{ Ans.}$$

In practice, much of the work can be done mentally and need not be written down.

2. Perform the indicated subtractions:

$$\frac{1}{a-b} - \frac{1}{2a} - \frac{1}{4b}.$$

The l. c. d. is $4ab(a-b)$. Reduce the fractions to the l. c. d.:

$$4ab(a-b) \div (a-b) = 4ab, \qquad \frac{1}{a-b} \cdot \frac{4ab}{4ab} = \frac{4ab}{4ab(a-b)}.$$

$$4ab(a-b) \div 2a = 2b(a-b), \qquad \frac{1}{2a} \cdot \frac{2b(a-b)}{2b(a-b)} = \frac{2ab - 2b^2}{4ab(a-b)}.$$

$$4ab(a-b) \div 4b = a(a-b), \qquad \frac{1}{4b} \cdot \frac{a(a-b)}{a(a-b)} = \frac{a^2 - ab}{4ab(a-b)}.$$

Subtracting and simplifying, $\dfrac{3ab + 2b^2 - a^2}{4ab(a-b)}$, *Ans.*

3. Perform the indicated additions:

$$\frac{3}{x^2-1} + \frac{2}{x(x+1)} + \frac{1}{1-x}.$$

The lowest common denominator is $x(x^2-1)$. Before reducing the last fraction to one with the denominator $x(x^2-1)$, it is a *convenience* to be able to write $x-1$ in place of $1-x$. As $1-x$ differs from $x-1$ only

in algebraic sign, we can do this, provided we change also the sign of the fraction or else change the sign of the numerator. The former change is simpler. Thus we have,

$$\frac{3}{x^2-1}+\frac{2}{x(x+1)}-\frac{1}{x-1}.$$

The reduction to the lowest common denominator, $x(x^2-1)$, is effected as follows:

$$x(x^2-1)\div(x^2-1)=x, \qquad \frac{3}{x^2-1}\cdot\frac{x}{x}=\frac{3x}{x(x^2-1)}.$$

$$x(x^2-1)\div x(x+1)=x-1, \qquad \frac{2}{x(x+1)}\cdot\frac{x-1}{x-1}=\frac{2x-2}{x(x^2-1)}.$$

$$x(x^2-1)\div(x-1)=x(x+1), \qquad \frac{1}{x-1}\cdot\frac{x(x+1)}{x(x+1)}=\frac{x^2+x}{x(x^2-1)}.$$

We obtain $\dfrac{3}{x^2-1}+\dfrac{2}{x(x+1)}-\dfrac{1}{x-1}=\dfrac{3x+2x-2-x^2-x}{x(x^2-1)}$

$$=\frac{4x-2-x^2}{x(x^2-1)}, \textit{ Ans.}$$

4. Perform the indicated addition and subtraction:

$$\frac{1}{(a-b)(c-a)}-\frac{1}{(a-b)(c-b)}+\frac{1}{(c-b)(a-c)}.$$

Here the factor $(a-b)$ occurs twice, and both times with the same order of the letters. The same is true of $(c-b)$. But in the first denominator occurs the factor $(c-a)$, in the last denominator occurs $(a-c)$. It is a *convenience* to have $(c-a)$ in both denominators. Since $(a-c)$ differs from $(c-a)$ only in sign, we write in the last denominator $(c-a)$ and at the same time change the sign of the fraction. We obtain

$$\frac{1}{(a-b)(c-a)}-\frac{1}{(a-b)(c-b)}-\frac{1}{(c-b)(c-a)}.$$

The l. c. d. $=(a-b)(c-a)(c-b)$. Reducing to a common denominator and adding,

$$\frac{c-b}{(a-b)(c-a)(c-b)}-\frac{c-a}{(a-b)(c-a)(c-b)}-\frac{a-b}{(a-b)(c-a)(c-b)}$$
$$=0, \textit{ Ans.}$$

EXERCISES

69. Perform the indicated operations:

1. $\frac{2x}{3}+\frac{3x}{4}$.

2. $\frac{5a}{y}-\frac{2b}{x}$.

3. $\frac{7}{x+1}-\frac{8}{x+1}$.

4. $\frac{6}{x-1}+\frac{5}{x+1}$.

5. $\frac{m}{x-y}-\frac{n}{x-y}$.

6. $\frac{a-b}{a+b}+\frac{a+b}{a-b}$.

7. $\frac{a-b}{ab}+\frac{b-c}{bc}+\frac{c-a}{ac}$.

8. $\frac{x+y}{(y-z)(z-x)}+\frac{y+z}{(z-x)(x-y)}+\frac{z+x}{(x-y)(y-z)}$.

9. $\frac{3}{1-2a}-\frac{7}{1+2a}+\frac{4-20a}{1-4a^2}$.

10. $\frac{1}{a+b}+\frac{1}{(a+b)^2}-\frac{1}{a^2-b^2}$.

11. $\frac{1}{2(x-y)}-\frac{1}{2(x+y)}-\frac{y}{x^2-y^2}$.

12. $\frac{3}{m+n}+\frac{-1}{m+3n}-\frac{3}{m-n}+\frac{1}{m-3n}$.

13. $\frac{a}{a^2-b^2}-\frac{b}{a^2+b^2}+\frac{a^3+b^3}{b^4-a^4}-\frac{-ab}{(a+b)(a^2+b^2)}$

14. $\frac{1+x}{(x-y)(x-z)}+\frac{1+y}{(y-z)(y-x)}+\frac{1+z}{(z-x)(z-y)}$.

15. $\frac{m+n}{(a-b)(a-c)}+\frac{n+s}{(b-c)(b-a)}-\frac{m+s}{(a-c)(c-b)}$.

16. $\dfrac{a^2+2a+4}{a+2}-\dfrac{a^2-2a+4}{2-a}+\dfrac{4-a^2}{a-2}.$

17. $\dfrac{1}{4a+4}+\dfrac{1}{4-4a}-\dfrac{1}{12-12a^2}+\dfrac{1}{3-3a}.$

18. $\dfrac{1}{x^2-x-6}-\dfrac{1}{x^2+6x+8}+\dfrac{1}{12-x-x^2}.$

19. $\dfrac{x-y-z}{(x-y)(x-z)}+\dfrac{y-z-x}{(y-z)(y-x)}+\dfrac{z-x-y}{(z-x)(z-y)}.$

20. $\dfrac{a}{p(p-q)(p-r)}-\dfrac{-a}{q(q-p)(q-r)}-\dfrac{a}{pqr}.$

21. $\dfrac{2}{2-a}+\dfrac{3}{a+3}-\dfrac{-10}{a^2+a-6}.$

22. $\dfrac{x}{x+1}+\dfrac{2x}{1-x^2}-\dfrac{-3x}{x-1}.$

23. $\dfrac{1}{2-x-x^2}+\dfrac{1}{x^2+3x+2}-\dfrac{1}{1-x^2}.$

24. $\dfrac{2}{(a-2)(a-3)}-\dfrac{3}{(3-a)(a-1)}-\dfrac{4}{(1-a)(2-a)}.$

25. $\dfrac{1}{(a-b)(b-c)}+\dfrac{1}{(c-a)(b-a)}+\dfrac{1}{(a-c)(b-c)}.$

QUADRATIC EQUATIONS

70. The equation $ax^2+bx+c=0$ is a *complete quadratic equation.* It is a *quadratic* equation because the highest power of the unknown x is the second; it is *complete* because it contains a term involving the unknown x to the first power and a term c (called the *absolute term*) which is free from x.

Quadratic equations of the forms

$$x^2 = c,$$
$$x^2 + bx = 0,$$

are called *incomplete quadratic equations*, because either the term involving x to the first power or the absolute term c is absent.

71. Incomplete quadratic equations are easily solved.

Take the form $x^2 = c.$
Extracting the square root of both sides of the equation, $x = \pm\sqrt{c}.$

The form $x^2 + bx = 0$ is solved by factoring, but may be solved also by the method of "completing the square," to be explained later.

Factoring, we obtain, $x(x + b) = 0,$
Make the first factor equal to zero, $x = 0.$
Make the second factor equal to zero, $x + b = 0.$
$x = -b.$

72. The solution of quadratic equations leads to *two* values of the unknown quantity. As both of these values are usually of interest and importance in the solution of problems, it is customary, in the extraction of square roots, to write down *both* results, the *principal* value and also the second value. This is indicated by the use of the symbol $\pm$. Thus, in "$x = \pm\sqrt{c}$," $+\sqrt{c}$ is the principal root, $-\sqrt{c}$ is the second root.

Since, in finding $x = \pm\sqrt{c}$, the square root of both sides of the equation has been extracted, it might be claimed that the sign $\pm$ should be written on *both* sides, giving

$$\pm x = \pm\sqrt{c}.$$

But this result is the same as when we write $x = \pm\sqrt{c}.$

For, the equation	$\pm x = \pm\sqrt{c}$	means here
(1) $+x = +\sqrt{c}.$		(3) $+x = -\sqrt{c}.$
(2) $-x = -\sqrt{c}.$		(4) $-x = +\sqrt{c}.$

Of these four sets, the first two are the same, and the last two are the same. Hence, $x = \pm\sqrt{c}$ gives all the values of x.

73. A *complete quadratic equation* may be solved in three ways:

(1) By *factoring,*

(2) By *completing the square,*

(3) By *substitution in a formula* obtained by the method of completing the square.

I. With our present knowledge of factoring, the first method is applicable only when the roots of the quadratic equation are rational. The method depends upon the principle that, *if the product of two or more factors equals zero, one factor must equal zero.*

Solve $$3x^2 = 2 + 5x.$$

Transpose *all* the terms to the first side, $3x^2 - 5x - 2 = 0.$

Factor, $(3x + 1)(x - 2) = 0.$

Place the first factor equal to zero, $3x + 1 = 0.$

$x = -\frac{1}{3}.$

Place the second factor equal to zero, $x - 2 = 0.$

$x = 2.$

The required roots are $2, -\frac{1}{3}$.

II. The second method depends on the type form of a trinomial which is a perfect square: $a^2 \pm 2ab + b^2$.

If the b^2 is lacking, we may take the square root of the first term, double it, divide the middle term by this, and square the quotient.

Take, for illustration, $x^2 \pm 5x$. To complete the square, we take the principal square root of x^2, which is x; double it, $2x$; divide $5x$ by $2x$, $\frac{5}{2}$; square $\frac{5}{2}$, $\frac{25}{4}$.

Hence $x^2 \pm 5x + \frac{25}{4}$ is a perfect square.

This results in the following rule for the solution of a quadratic equation:

1. *Transpose all terms containing x^2 and x to the left side of the equation; all others to the right side.*

2. *Divide both sides by the coefficient of x^2.*

3. *Add to both sides the* **square of half the coefficient of x.**

4. *Extract the square root of both sides and solve the resulting equations.*

Solve $$3x^2 - 5x - 2 = 0.$$

Transpose, $$3x^2 - 5x = 2.$$
Divide by 3, $$x^2 - \tfrac{5}{3}x = \tfrac{2}{3}.$$
Add $(\frac{1}{2} \cdot \frac{5}{3})^2$, $$x^2 - \tfrac{5}{3}x + \tfrac{25}{36} = \tfrac{49}{36}.$$
Take the square root, $$x - \tfrac{5}{6} = \pm\tfrac{7}{6}.$$
Whence, $$x = \tfrac{5}{6} \pm \tfrac{7}{6}.$$
$$x = 2, -\tfrac{1}{3}.$$
Ans., $2, -\frac{1}{3}$.

III. The third method depends upon the formula derived from the solution of the type form of the complete quadratic.

Solve $$ax^2 + bx + c = 0.$$

Transpose c, $$ax^2 + bx = -c.$$
Divide by a, $$x^2 + \frac{b}{a}x = -\frac{c}{a}.$$
Add $\left(\frac{1}{2} \cdot \frac{b}{a}\right)^2$, $$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2 - 4ac}{4a^2}.$$
Take the square root, $$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a},$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

In using the formula, transpose *all* terms of the given equation to the left side. Why?

Solve $$3x^2 - 5x = 2.$$

Transpose the 2, $$3x^2 - 5x - 2 = 0.$$
Here $a = 3$, $b = -5$, $c = -2$.
$$\therefore x = \frac{5 \pm \sqrt{25 + 24}}{6} = \frac{5 \pm 7}{6} = 2 \text{ or } -\tfrac{1}{3}.$$

EXERCISES

74. Solve; if a numerical equation has irrational roots, *approximate* their values, to three decimal places. Use the table of square roots in § 197..

1. $x^2 + 4x = 5$.
2. $3x^2 + x - 14 = 0$.
3. $x^2 - 8x = -11$.
4. $x^2 - 2x - 15 = 0$.

5. $x^2 + 6x = -9.$

6. $x^2 + x + 1 = 0.$

7. $x^2 - ax = c.$

8. $bx^2 - cx = d.$

9. $3x^2 + x + 7 = 0.$

10. $x^2 - 2ax = b^2 - a^2.$

11. $m^2x^2 - 2mx = n - m^2.$

12. $ax + \dfrac{ab}{b+c} = (b+c)x^2.$

13. $(3x - 5)^2 = 4x.$

14. $12x^2 + 13x - 35 = 0.$

15. $\dfrac{1}{x-1} + \dfrac{1}{x-2} = -\dfrac{1}{x-3}.$

16. $(x+2)(x-2) = 7(x+2) - 5.$

17. $(x+3)^2 - 2(x+3) + 1 = 0.$

18. $5x(x-3) - 2(x^2-6) = (x+3)(x+4).$

19. $1 - \dfrac{x+2}{2x+1} = \dfrac{x-5}{x-3}.$

20. $\dfrac{m^2}{(x+m)^2} = \dfrac{n^2}{(x-n)^2}.$

21. $\dfrac{(c^2-d^2)(y^2+1)}{c^2+d^2} = 2y.$

22. $\dfrac{(2y-a)^2}{2y-a+2c} = c.$

23. $(x-1)(x+2)(x^2+x-1) = 0.$

24. $(x^2-7x+12)(x^2+3x+2) = 0.$

PROBLEMS

Ascertain in each problem whether both roots of the quadratic equation are applicable to the problem.

75. 1. Find two consecutive numbers whose product is 992.

2. Find the length and breadth of a rectangle whose area is 375 sq. in., and whose length exceeds its breadth by 10 in.

3. Express 71 as the sum of two numbers whose product is 448.

4. If the length of a square be increased by 2, and the width be increased by 3, the area of the resulting rectangle is 40. Determine the length of a side of the square.

5. The height of a triangle exceeds its base by 8; if the area of the triangle is 1209, what is its base?

6. The diagonal of a square is 2 ft. longer than the side. Find the side.

7. A cylinder 12 ft. in height has a capacity 125 cu. ft. Determine the diameter of its base.

8. When the edges of a cube are each increased by 6 in., the volume is increased by 936 cu. in. Find the dimensions of the original cube.

9. The sum of the numerator and denominator of a fraction is 77. If the numerator is increased by 111 and the denominator is increased by 40, the fraction is doubled. Find the fraction.

10. Find two numbers which differ by 2, the cubes of which differ by 296.

11. The radius of one circle is twice the radius of another. Find the radii of both, if the difference of their areas is 75.

12. The difference of the volumes of two spheres is 100 cu. in.; the difference of their radii is 5 in. Find their radii, correct to two decimal places.

The volume of a sphere is $\frac{4\pi r^3}{3}$.

13. A woman paid \$64 for silk. If she had bought 4 yards less for the same money, she would have paid \$$1\frac{1}{3}$ more per yard. How many yards did she buy?

14. The longer leg of a right triangle exceeds the shorter leg by 3 ft. The area of the triangle is 135 sq. ft. Find the length of each leg.

15. A bookdealer sells a number of algebras for \$87. Had he reduced the price of each book by 12¢, he would have sold 16 more books for the same sum of money. How many books would he have sold at the reduced rate?

16. If a man's daily wage had been \$1 less, it would have taken him 15 days longer to earn \$180. How many days did he work to earn \$180?

17. A picture $10'' \times 14''$ is placed in a frame of uniform width. If the area of the frame is equal to half the area of the picture, how wide is the frame?

Draw a figure.

18. Find two consecutive even numbers whose product is 528.

19. Find two consecutive odd numbers whose product is 8099.

20. If the difference between the parallel sides of a trapezoid is 5 ft., and the altitude of the trapezoid is equal to the longer of the parallel sides, find the lengths of the parallel sides when the area is 2375 sq. ft.

21. A flower bed is $15' \times 20'$. How wide a walk must surround the bed, to increase the total area by 770 sq. ft.?

22. A tinner makes a square box 3 in. deep, with a capacity of 1587 cu. in. From each corner of a square sheet of tin a 3-inch square is cut and the four rectangular parts of the tin are turned up. What are the dimensions of the square sheet of tin?

Draw a figure.

23. If a square has its length reduced by 7 in. and its width by 10 in., what are the linear dimensions of the resulting rectangle, if its area is 8370 sq. in.?

24. An oil tank can be filled by one pipe in 2 hours less time than by another pipe. If both pipes are open $1\frac{1}{2}$ hours, the tank will be filled. In what time can the tank be filled by each pipe?

25. A number of postage stamps can be arranged in a rectangle, each side containing 60 stamps. If the same number

of stamps be arranged in two rectangles so that each side of one rectangle will contain 12 more stamps than each side of the other, how many stamps does a side of each of the latter rectangles contain?

26. A boat's crew can row at the rate of 9 miles an hour. What is the speed of the current in the river if it takes them 2 hours and 15 minutes to row 9 miles up stream and back?

27. Divide $ 1248 among three persons, so that the second shall have $ 3 more than the first, and the third shall have as many times the share of the second as there are dimes in the first person's share.

28. The population of a city increases from 20,000 to 20,808 in two years. What is the annual rate of increase per hundred?

29. A sum of $ 2000 drawing interest that is compounded annually, amounts to $ 2142.45 in two years. Find the rate of interest.

CHAPTER II

MORE ADVANCED THEORY AND OPERATIONS

FUNDAMENTAL LAWS OF ALGEBRA

76. The operations of algebra obey certain fundamental laws which we have not formulated thus far. Nevertheless we have so accustomed ourselves to follow them, that we find it difficult to see how a new algebra might be made, in which a different set of laws would prevail. We shall now explain the laws which underlie our algebra.

If several positive and negative numbers are added or subtracted, it matters not in what order the operations are performed; the numbers may be *commuted* at pleasure.

For example, $4 + 7 - 6 = 5,$
or $7 + 4 - 6 = 5,$
or $-6 + 4 + 7 = 5.$

This is called the *commutative law for addition.** Using letters, the law may be stated thus,

$$a + b = b + a.$$

* In our algebra, addition and subtraction may be represented geometrically by the addition and subtraction of distances along a straight line. Let a and b represent distances measured off toward the right, then $a + b$ and $b + a$ both represent the same distance OC in Fig. 3; the commutative law is obeyed.

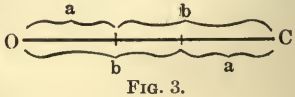

FIG. 3.

Suppose, now, that a and b are assigned meanings entirely different from those given above; suppose a means a *rotation about the line OA* as an axis, through 90 degrees, and b means a *rotation about OB* as an axis, through 90 degrees.

Again, $5 + 9 - 2 = 12,$

or $(5 + 9) - 2 = 12,$

or $5 + (9 - 2) = 12.$

That is, the final result is the same, whether the 9 be associated with the 5 or with the -2.

This is called the *associative law for addition.* Using letters, it may be expressed thus,

$$a + b + c = (a + b) + c$$
$$= a + (b + c).$$

If an expression contains two or more factors, it matters not in what order the multiplications are performed.

For example, $2 \cdot 7 = 7 \cdot 2.$

This is called the *commutative law for multiplication.** Using letters, the law may be expressed thus:

$$a \cdot b = b \cdot a.$$

Let a rectangle be the figure rotated (Fig. 4). Then $a + b$ (*i.e.* the rotation about OA, followed by a rotation about OB) brings the rectangle in a position where "Alg." is horizontal, as in Fig. 5. On the other hand, $b + a$ (*i.e.* a rotation about OB, followed by a rotation about OA) brings the rectangle in

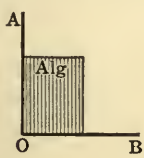

FIG. 4.

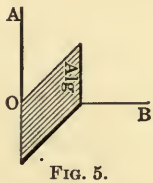

FIG. 5.

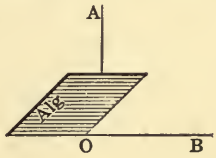

FIG. 6.

a position in which "Alg." is vertical (Fig. 6). Since the final positions of the rectangle are different, it follows that in this case, $a + b \neq b + a$. That is, the commutative law is *not* obeyed. (The symbol $\neq$ means "is not equal to.")

Whether the commutative law for addition is obeyed or not depends therefore upon the definitions given to a and b, and to the processes of addition and subtraction.

* There is an advanced algebra, called *quaternions*, in which $ij \neq ji$; that is, the commutative law for multiplication does not generally hold true. Quaternions are used in the study of mathematical physics.

Again, it matters not how the factors are associated or grouped, for

$$6 \cdot 5 \cdot 3 = 90.$$
$$6 \cdot (5 \cdot 3) = 90,$$
$$(6 \cdot 5) \cdot 3 = 90.$$

This is called the *associative law for multiplication.* Using letters, the law may be expressed thus:

$$a \cdot b \cdot c = a \cdot (b \cdot c) = (a \cdot b) \cdot c.$$

Again, a factor placed before or after a parenthesis containing two or more terms may be distributed among the various terms without any change in the final result.

That is, $$5(9 - 4 + 6) = 5 \times 9 - 5 \times 4 + 5 \times 6 = 55.$$

This is called the *distributive law for multiplication.* Using letters, the law may be expressed thus:

$$a(b + c) = ab + ac.$$

HISTORICAL NOTE

77. It is a curious fact that, in the development of algebra, the fundamental laws were the last things to be explained. The beginnings of algebra can be traced back to about 2000 years before Christ, but not until the nineteenth century were the fundamental laws of algebra formulated. In the earlier treatment the laws were tacitly assumed to be true. For instance, it was assumed that $ab = ba$, $a(b + c) = ab + ac$, $(ab)c = a(bc)$, without special attention being called to this matter nor special names being given to the relations assumed. The need of an explicit statement of the fundamental laws came to be recognized when it was perceived that, besides the algebra which we are studying, in which, for example, ab is always equal to ba, there could be established other algebras in which ab is not always equal to ba. Among those who helped to perfect the science of algebra along these lines were the Englishmen, George Peacock, D. F. Gregory, Augustus De Morgan, and Sir William Rowan Hamilton; the Frenchmen, F. J. Servois, and A. L. Cauchy; the Germans, Martin Ohm, Hermann Grassmann, and Hermann Hankel; and the American, Benjamin Peirce. The names "commutative law," "distributive law," were first used by Servois in 1814. Among the first to use the name "associative law" was Sir William Rowan Hamilton.

AUGUSTUS DE MORGAN (1806–1871)

Was professor of mathematics in London and wrote papers on the foundations of algebra and on logic. To an English journal, the *Athenæum*, he contributed amusing articles on circle squarers. These articles were afterwards collected in a book, entitled *A Budget of Paradoxes*.

REMAINDER THEOREM AND FACTOR THEOREM

78. Divide $x^3 - 5x^2 + 7x - 2$ by $x - a$.

$$
\begin{array}{l|l}
x^3 - 5x^2 + 7x - 2 & x - a \\
\underline{x^3 - ax^2} & \overline{x^2 + (a-5)x + (a^2 - 5a + 7)} \\
(a-5)x^2 + 7x - 2 & \\
\underline{(a-5)x^2 - (a^2 - 5a)x} & \\
\qquad (a^2 - 5a + 7)x - 2 & \\
\qquad \underline{(a^2 - 5a + 7)x - a^3 + 5a^2 - 7a} & \\
\qquad\qquad a^3 - 5a^2 + 7a - 2 &
\end{array}
$$

Observe that the remainder, $a^3 - 5a^2 + 7a - 2$, is the same as the dividend, if in the dividend a is substituted for x. This illustrates the

Remainder Theorem: *If a rational integral expression in x is divided by $x - a$, the remainder is the same as the original expression with a substituted for x.*

Now $x^3 - 5x^2 + 7x - 2$ would be exactly divisible by $x - a$, if $a^3 - 5a^2 + 7a - 2 = 0$.

Hence such a polynomial will be exactly divisible by a binomial of the form $x - a$, if, when a is substituted for x, the polynomial vanishes, *i.e.* equals zero. This illustrates the

Factor Theorem: *If a rational integral expression in x becomes zero when a number a is substituted for x, then $x - a$ is a factor of the expression.*

Thus, the expression $x^6 + 2x^5 - x - 2$, for $x = 1$, becomes $1 + 2 - 1 - 2 = 0$, hence $x - 1$ is a factor of the expression.

Again, for $x = -2$, this expression becomes $64 - 64 + 2 - 2 = 0$, hence $x - (-2)$ or $x + 2$ is a factor of it.

A third factor can be found by dividing $x^6 + 2x^5 - x - 2$ by the product of $x - 1$ and $x + 2$; that is, by $x^2 + x - 2$. The quotient is $x^4 + x^3 + x^2 + x + 1$. Hence we have,

$$x^6 + 2x^5 - x - 2 = (x - 1)(x + 2)(x^4 + x^3 + x^2 + x + 1).$$

In the example above we took $x = 1$ and $x = -2$. Notice that 1 and -2 are both integral *factors* of the absolute term

-2 in the expression $x^6 + 2x^5 - x - 2$. There is a theorem bearing on this point which we shall use along with the Factor Theorem. It relates to *rational integral expressions,*

$$x^n + ax^{n-1} + bx^{n-2} + \cdots + hx + k.$$

The theorem, which we give without proof, is as follows:

When a rational integral expression has 1 *as the coefficient of the highest power of x, and the other coefficients $a, b, \cdots, h, k$ are all integral numbers (either positive or negative), then, in searching for rational values of x that will make the given expression zero, only integral factors of the absolute term k need be tried.*

Factor $x^4 + 6x^3 + 42x - 49$.

The coefficient of x^4 is 1; the other coefficients, 6, 42, -49 are integers. Hence both conditions are satisfied, and we need to try only integral factors of 49; namely, ± 1, ± 7, ± 49.

When $x = 1$, $1 + 6 + 42 - 49 = 0$. Hence, by the Factor Theorem, $x - 1$ is a factor of the expression.

When* $x = -1$, $1 - 6 - 42 - 49 \neq 0$; $x + 1$ is *not* a factor.

When $x = 7$, $2401 + 2058 + 294 - 49 \neq 0$, $x - 7$ is *not* a factor.

When $x = -7$, $2401 - 2058 - 294 - 49 = 0$, $x + 7$ is a factor.

In the same way, $x - 49$ and $x + 49$ are found not to be factors.

Dividing the expression by $(x - 1)(x + 7)$ yields another factor, $x^2 + 7$.

Hence, $x^4 + 6x^3 + 42x - 49 = (x - 1)(x + 7)(x^2 + 7)$.

EXERCISES

79. 1. Divide $3x^4 - 7x^3 + 5x^2 - x - 1$ by $x - b$, then compare the remainder with the dividend. How does this illustrate the Remainder Theorem?

2. Divide $5x^5 - 4x^3 + 6x + 2$ by $x - a$ and illustrate the Remainder Theorem.

3. Using the Factor Theorem, factor

$$x^4 - 5x^3 + 9x^2 - 15x + 18.$$

4. Factor $x^3 - 17x^2 + 3x + 54$.

* The symbol $\neq$ means "is not equal to." See § 76, footnote.

5. What is the remainder when $2x^3 - 3x^2 + 5x - 1$ is divided by $x - c$? By $x - b$? By $x - 1$?

6. What is the remainder when $5x^3 + 2x^2 + 3x + 6$ is divided by $x + n$? By $x + 1$?

7. Is $x + 2$ an exact divisor of $x^4 - x - 6$?

8. If a rational integral expression, with 1 as the coefficient of the highest power of x, and with all other coefficients integers, is exactly divisible by $x - a$, where a is an integer, what relation is a to the last term?

9. Find a factor of $x^5 - x^4 + x^3 - x^2 + x - 1$.

10. Find a factor of $x^3 - 9x^2 + 17x - 6$.

11. Find a factor of $2x^4 - x^3 - 2x^2 - 3x - 10$.

12. Find a factor of $x^3 - 13x + 12$.

13. Factor $x^5 - a^5$.

If a is substituted for x, then $a^5 - a^5 = 0$. Hence, by the Factor Theorem, $x - a$ is a factor of $x^5 - a^5$. Division of $x^5 - a^5$ by $x - a$ yields a second factor. Hence

$$x^5 - a^5 = (x - a)(x^4 + x^3a + x^2a^2 + xa^3 + a^4).$$

The quotient may be obtained by the following special method of short division:

$x^5 \div x = x^4.$ $\quad x^4 \div x \cdot a = x^3a.$ $\quad x^3a \div x \cdot a = x^2a^2.$

$x^2a^2 \div x \cdot a = xa^3.$ $\quad xa^3 \div x \cdot a = a^4.$

14. Factor $x^5 + a^5$.

If $-a$ is substituted for x, then $(-a)^5 + a^5 = -a^5 + a^5 = 0$. Hence by the Factor Theorem, $x - (-a)$ or $x + a$ is a factor of $x^5 + a^5$. By division, we find a second factor. Hence

$$x^5 + a^5 = (x + a)(x^4 - x^3a + x^2a^2 - xa^3 + a^4).$$

The special rule of division given in Ex. 13 applies to this case, except that the signs in the quotient are alternately + and −.

15. Factor $x^6 + a^6$.

If $\pm a$ is substituted for x, we obtain $(\pm a)^6 + a^6 = a^6 + a^6 \neq 0$.

Hence, neither $x + a$ nor $x - a$ is a factor of $x^6 + a^6$. But $x^6 + a^6$ may be considered as the sum of two cubes $(x^2)^3 + (a^2)^3$ and can therefore be factored, as $y^3 + z^3$ is factored.

16. Explain the following results pertaining to *rational* and *real* factors:

$a^2 + b^2$ cannot be factored.

$a^4 + b^4$ cannot be factored.

$a^6 + b^6 = (a^2 + b^2)(a^4 - a^2b^2 + b^4)$.

$a^8 + b^8$ cannot be factored.

NOTE. The sum of the same two *even* powers of two numbers may be factored, *if* the exponent is the product of an *odd* and an *even* factor. One factor is the sum of the numbers with exponents equal to the *even* factor; the other factor may be found by inspection or by long division. Thus, in $a^{12} + b^{12}$, the exponent $12 = 3 \cdot 4$, where 3 is odd, 4 even.

One factor of $a^{12} + b^{12}$ is $a^4 + b^4$. The other factor is $a^8 - a^4b^4 + b^8$. Hence, $a^{12} + b^{12} = (a^4 + b^4)(a^8 - a^4b^4 + b^8)$.

If numerical cubic equations of the form $x^3 + ax^2 + bx + c = 0$, a, b, c being positive or negative integers, have at least one root *rational*, then the equation can be solved with the aid of the Factor Theorem.

$x^3 + 4x^2 - 2x - 5$ becomes zero when -1 is substituted for x. Hence $x + 1$ is a factor, the quotient $x^2 + 3x - 5$ is another factor.

Write the cubic equation in the form $(x + 1)(x^2 + 3x - 5) = 0$.

Make the first factor equal to zero, $x + 1 = 0$,

and $x = -1$.

Make the second factor equal to zero, $x^2 + 3x - 5 = 0$,

and solve the quadratic equation, $x = \dfrac{-3 \pm \sqrt{29}}{2}$.

Hence the roots of the cubic are -1, $\dfrac{-3 + \sqrt{29}}{2}$, $\dfrac{-3 - \sqrt{29}}{2}$.

17. Solve $(x - 1)(x^2 + 6x + 8) = 0$.

Solve the following cubic equations:

18. $x^3 + 5x^2 + 7x + 3 = 0$.

19. $x^3 + 5x^2 - 3x - 22 = 0$.

20. $x^3 - 6x^2 - 8x + 7 = 0$.

21. $x^3 + 18x^2 + 82x + 55 = 0$.

ORAL REVIEW

80. Factor:

1. $x^2 - 16y^2$.
2. $25a^2 - b^2$.
3. $4x^2 - 9y^2$.
4. $x^3 + y^3$.
5. $x^3 - y^3$.
6. $x^3 - 1$.
7. $1 + y^3$.
8. $ax^2 - ay^2$.
9. $x^2 - 100$.
10. $x^3 - 1000$.
11. $m^3 + 8$.
12. $m^3 - 8$.
13. $27 - a^3$.
14. $27 + a^3$.
15. $m^5 + n^5$.
16. $m^5 - n^5$.
17. $a^4 - b^4$.
18. $a^4 + b^4$.
19. $y^4 - 1$.
20. $y^4 - y^3$.
21. $x^4 + x^2$.
22. $x^4 - x^2$.
23. $10x^3 - 10$.
24. $10x^3 + 10$.
25. $x^3 + 3x^2 + 3x + 1$.
26. $y^3 - 3y^2 + 3y - 1$.
27. $x^4 + 2x^2y^2 + y^4$.
28. $x^3 + 8y^3$.
29. $z^3 - 27w^3$.
30. $x^4 - 16y^4$.
31. $10x^2 + 20xy + 10y^2$.
32. $x^6 - 1$.
33. $x^6 + 1$.
34. $x^2 - (2y + z)^2$.
35. $x^2 - (2y - z)^2$.
36. $2a^5 - 2a$.

WRITTEN REVIEW

81. Factor:

1. $8a^3 + b^3$.
2. $c^{12} - d^{12}$.
3. $c^{12} + d^{12}$.
4. $27x^3 - 125y^3$.
5. $27x^3 - y^6$.
6. $8m^3 + 27n^3$.
7. $27x^3 - 64y^3$.
8. $m^5 - n^5p^5$.
9. $a^8 - b^8$.
10. $c^6 + 64d^6$.
11. $x^4 - (2x - y)^4$.
12. $x^3 + (y + z)^3$.
13. $x^4 + 4x^3 - 2x^2 - 4x + 1$.
14. $x^5 - 4x^4 - 7x^2 + 28$.
15. $x^4 - 3x^3 + 7x^2 - 13x + 6$.
16. $x^3 - ax^2 - 5a^2x + 5a^3$.
17. $m^3 - 6m^2 + 12m - 8$.
18. $x^{10} - y^{10}$.
19. $x^4 + x^2y^2 + y^4$.

    Hint. $x^4 + x^2y^2 + y^4$
    $= x^4 + 2x^2y^2 + y^4 - x^2y^2$.
20. $a^4 + a^2 + 1$.
21. $4x^4 + 3x^2y^2 + 9y^4$.
22. $a^4 - 7a^2b^2 + b^4$.
23. $49p^4 + 68p^2q^2 + 36q^4$.
24. $64x^4 + 55x^2y^2 + 25y^4$.

IRRATIONAL AND IMAGINARY FACTORS

82. Thus far all factoring has been confined to the discovery of *rational* factors. But it is sometimes advantageous to resolve expressions into factors which involve irrational or even imaginary terms. Expressions like $a^2 - 2b^2$, $a^4 - 3b^4$, $a^2 + b^2$, or $a^4 + b^4$ can be factored, if the restriction is removed that the factors must be real and rational.

EXERCISES

83. 1. Factor $a^2 - 3$.

This may be done by the type form $a^2 - b^2 = (a + b)(a - b)$, where $b^2 = 3$ and $b = \sqrt{3}$. We obtain

$$a^2 - 3 = (a + \sqrt{3})(a - \sqrt{3}).$$

2. Factor $x^2 + y^2$.

This, too, may be factored by the type form $a^2 - b^2 = (a + b)(a - b)$. Write $x^2 + y^2$ in the form $x^2 - (-y^2)$. Since $\sqrt{-y^2} = y\sqrt{-1} = iy$, where $i = \sqrt{-1}$, we have,

$$x^2 + y^2 = (x + iy)(x - iy).$$

Resolve into irrational or imaginary factors:

3. $m^2 - 5$.
4. $a^2 - 12$.
5. $a^2 + 4$.
6. $m^2 + 9n^2$.
7. $a^4 + b^2$.
8. $r^2 - 5s^2$.
9. $2a^2 - b^2$.
10. $x^2 - 8y^2$.
11. $9a^2 + 16b^2$.
12. $a^4 - 3b^4$.
13. $a^2 - b$.
14. $a^4 - 4b^4$.

15. By factoring, find all four roots of the equation $x^4 - 1 = 0$.

16. By factoring, find all four roots of the equation $x^4 - 16 = 0$.

THE BINOMIAL THEOREM

84. By multiplication, we find that

$$(a \pm b)^1 = a \pm b.$$

$$(a \pm b)^2 = a^2 \pm 2\,ab + b^2.$$

$$(a \pm b)^3 = a^3 \pm 3\,a^2b + 3\,ab^2 \pm b^3.$$

$$(a \pm b)^4 = a^4 \pm 4\,a^3b + 6\,a^2b^2 \pm 4\,ab^3 + b^4.$$

$$(a \pm b)^5 = a^5 \pm 5\,a^4b + 10\,a^3b^2 \pm 10\,a^2b^3 + 5\,ab^4 \pm b^5.$$

$$(a \pm b)^6 = a^6 \pm 6\,a^5b + 15\,a^4b^2 \pm 20\,a^3b^3 + 15\,a^2b^4 \pm 6\,ab^5 + b^6,$$

etc., wherein a and b represent the first and second terms, respectively, of any binomial.

A careful study of these products will show that they follow certain laws, by which they may be written down without recourse to laborious multiplications.

In the products we observe the following laws:

I. *The first term is a raised to the same power as that of the binomial. In each succeeding term the exponent of a decreases by* 1.

II. *The factor b does not appear in the first term. The exponent of b in the second term is* 1 *and increases by* 1 *in each succeeding term.*

III. *The coefficient of the first term is* 1. *The coefficient of any term after the first is found by multiplying the coefficient of the preceding term by the exponent of a in that term, and dividing by* 1 *more than the exponent of b.*

IV. *If the binomial is $a + b$, the signs in the product are all plus; if the binomial is $a - b$, the signs are alternately* + *and* −.

V. *The number of terms is one more than the exponent of the binomial.*

VI. *Each term is of the same degree as the binomial.*

Using these laws, find without actual multiplication the product of $(a+b)^7$.

The first term is by I, a^7.

The second term has by III the coefficient $\frac{1 \times 7}{1}$ or 7, by I the factor a^6, by II the factor b. Hence the term is $+7\,a^6b$.

The third term has by III the coefficient $\frac{7 \times 6}{2}$ or 21, by I the factor a^5, by II the factor b^2. Hence the term is $+21\,a^5b^2$.

The fourth term has by III the coefficient $\frac{21 \times 5}{3}$ or 35, by I the factor a^4, by II the factor b^3. Hence the term is $+35\,a^4b^3$, and so on. By V the final result has 8 terms. We obtain,

$$(a+b)^7 = a^7 + 7\,a^6b + 21\,a^5b^2 + 35\,a^4b^3 + 35\,a^3b^4 + 21\,a^2b^5 + 7\,ab^6 + b^7.$$

All terms are of the seventh degree, as required by VI.

85. If n is a positive integer, then the products found above may all be represented by one general formula, as follows:

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{2 \cdot 3}a^{n-3}b^3 + \cdots + nab^{n-1} + b^n.$$

This formula is called the *Binomial Theorem.* By actual multiplication it was shown above to be true for all positive integral values of n, up to 6. We found the expansion of $(a+b)^7$ on the tacit assumption that the formula holds for $n=7$. Really we had no right to take for granted, without proof, that the laws given above do hold true for $n>6$. Frequently certain relations hold true up to a certain point, but no further.

For example, the first three integers 1, 2, 3 are all prime numbers. A careless reasoner might be tempted to jump to the conclusion that all integers are prime numbers, which is, of course, not true.

That the Binomial Theorem is true for any positive integral value of n will be proved later in § 193.

Expand $(2x - 3y)^4$.

Here $a = 2x$, $b = 3y$. For convenience in expanding, place $2x$ and $3y$ in parentheses, thus $(2x)$ and $(3y)$. By Law IV of § 84, the terms in the expansion are alternately $+$, $-$. We obtain,

$$(2x - 3y)^4 = (2x)^4 - 4(2x)^3(3y) + 6(2x)^2(3y)^2 - 4(2x)(3y)^3 + (3y)^4.$$

Simplify each term in the expansion:

$$(2x - 3y)^4 = 16x^4 - 96x^3y + 216x^2y^2 - 216xy^3 + 81y^4.$$

EXERCISES

86. Expand:

1. $(x + y)^6$.
2. $(a - b)^8$.
3. $(2x + y)^3$.
4. $(a - 3b)^3$.
5. $(c^2 - d^2)^4$.
6. $(3a - b^2)^5$.
7. $(ax - y)^{10}$.
8. $(3a^2 + 2b^3)^3$.
9. $(m - 2n)^7$.
10. $(2r^3 - 3s^2)^4$.
11. Write the first three terms of $(a - b)^{20}$.
12. Write the first five terms of $(x + y)^{35}$.
13. Compare the coefficients of the first and last terms in the given expansions in § 84; of the terms next to the first and last.
14. What is the third term of $(mx + ny)^{10}$?
15. Write the 9th term of $(2a^2 - 3b)^9$.

SYSTEMS OF LINEAR EQUATIONS. DETERMINANTS

87. Systems of n independent linear equations in n unknowns may be solved by eliminating the same unknown from all the equations. The resulting system contains in general one less equation and one less unknown.

From this resulting system eliminate a second unknown, and continue this process until one equation containing only one unknown is obtained.

EXERCISES

88. 1. Solve the system of equations

$$x + y + z = 9, \qquad (1)$$
$$x + 2y - z = 8, \qquad (2)$$
$$2x - 3y + 4z = 7. \qquad (3)$$

Solution. Subtract (2) from (1), $-y + 2z = 1.$ (4)

Subtract (3) from $2 \cdot (1)$, $5y - 2z = 11.$ (5)

Add (4) and (5), $4y = 12,$

$y = 3.$

Substitute 3 for y, in (4), $z = 2.$

Substitute 2 for z, and 3 for y, in (1), $x = 4.$

Check by substituting the answers in *all three* equations (1), (2), and (3).

2. $x - y + z = 20,$
$2x - y - z = -30,$
$3x - 4y + 2z = 10.$

3. $5x - 6y + 7z = 105,$
$3x - 6y + 8z = 103,$
$3y - 4z = -44.$

4. $x + y + z + w = 10,$
$x - y + 2z - 3w = -3,$
$3x - 2y + z - w = 5,$
$3x - 3y + 2z - 4w = -3.$

89. In previous parts, three methods of elimination have been explained: (1) by addition or subtraction, (2) by substitution, (3) by comparison.

A fourth method of solution is by means of a formula, called a *determinant.* In deriving this formula we use the old methods of solution.

Solve:
$$ax + by = c \qquad (1)$$
$$dx + ey = f. \qquad (2)$$

Multiply (1) by d, (2) by a, $adx + bdy = cd$ (3)

$adx + aey = af.$ (4)

Subtract (3) from (4), $(ae - bd)y = af - cd$

$$y = \frac{af - cd}{ae - bd}.$$

Similarly, $$x = \frac{ce - bf}{ae - bd}.$$

The numerator $af - cd$ may be written $\begin{vmatrix} a & c \\ d & f \end{vmatrix}$.

The term af is the product of a and f which lie on the diagonal from the upper left corner to the lower right corner.

The term cd is the product of c and d which lie on the diagonal from the upper right corner to the lower left corner.

The second product is *subtracted* from the first.

Similarly, $$\begin{vmatrix} 3 & 4 \\ 5 & 6 \end{vmatrix} = 3 \times 6 - 4 \times 5.$$

The numerator $ce - bf$ may be written $\begin{vmatrix} c & b \\ f & e \end{vmatrix}$.

The denominator $ae - bd$ may be written $\begin{vmatrix} a & b \\ d & e \end{vmatrix}$.

Expressions written in this form are called *determinants.*

Hence, $$x = \frac{\begin{vmatrix} c & b \\ f & e \end{vmatrix}}{\begin{vmatrix} a & b \\ d & e \end{vmatrix}}, \tag{5}$$

$$y = \frac{\begin{vmatrix} a & c \\ d & f \end{vmatrix}}{\begin{vmatrix} a & b \\ d & e \end{vmatrix}}. \tag{6}$$

The determinants in (5) and (6) can be used conveniently as formulas for solving two linear equations in two unknowns.

The determinants can be written down at once, according to the following rules:

I. *In the two denominators the determinant is the same. It is formed by writing the coefficients of x and y as they stand in the equations* (1) *and* (2).

II. *The determinant in the numerator of* (5) *is formed from the denominator by replacing* $\begin{matrix} a \\ d \end{matrix}$, *the coefficients of x, by the absolute terms* $\begin{matrix} c \\ f \end{matrix}$.

III. *The determinant in the numerator of* (6) *is formed by replacing* $\begin{matrix} b \\ e \end{matrix}$, *the coefficients of* y, *by the absolute terms,* $\begin{matrix} c \\ f \end{matrix}$.

Example. Solve $\begin{cases} 3x + 4y = 34, \\ 5x - 3y = -11. \end{cases}$

The denominator is $\begin{vmatrix} 3 & 4 \\ 5 & -3 \end{vmatrix}$.

The numerator for the value of x is $\begin{vmatrix} 34 & 4 \\ -11 & -3 \end{vmatrix}$.

The numerator for the value of y is $\begin{vmatrix} 3 & 34 \\ 5 & -11 \end{vmatrix}$.

Hence,

$$x = \frac{\begin{vmatrix} 34 & 4 \\ -11 & -3 \end{vmatrix}}{\begin{vmatrix} 3 & 4 \\ 5 & -3 \end{vmatrix}} = \frac{34 \cdot (-3) - 4 \cdot (-11)}{3(-3) - 4 \cdot 5} = \frac{-102 + 44}{-9 - 20} = \frac{-58}{-29} = 2,$$

$$y = \frac{\begin{vmatrix} 3 & 34 \\ 5 & -11 \end{vmatrix}}{\begin{vmatrix} 3 & 4 \\ 5 & -3 \end{vmatrix}} = \frac{3(-11) - 34 \cdot 5}{-29} = \frac{-33 - 170}{-29} = \frac{-203}{-29} = 7.$$

EXERCISES

90. Find the values of the following determinants:

1. $\begin{vmatrix} 9 & 6 \\ 8 & 7 \end{vmatrix}$

2. $\begin{vmatrix} .9 & .2 \\ .5 & .1 \end{vmatrix}$

3. $\begin{vmatrix} 1\frac{1}{3} & 2\frac{1}{4} \\ 1\frac{1}{2} & 1\frac{3}{4} \end{vmatrix}$

4. $5\begin{vmatrix} 3 & -2 \\ 4 & 10 \end{vmatrix}$

5. $\begin{vmatrix} 9 & 0 \\ a & b \end{vmatrix}$

6. $\begin{vmatrix} 7a & 8b \\ 4b & -3a \end{vmatrix}$

7. $\begin{vmatrix} a^3 & b^2 \\ a^{-1} & b^{-1} \end{vmatrix}$

8. $a\begin{vmatrix} bc & de \\ ae & -bc \end{vmatrix}$

9. $\begin{vmatrix} a^{-2} & b^{-3} \\ c^{-1} & d^{-4} \end{vmatrix}$

Solve, by determinants:

10. $2x + 3y = 4,$
$5x + 6y = 7.$

11. $a_1x + b_1y = c_1,$
$a_2x + b_2y = c_2.$

12. $2x - 5y = -13,$
$3x - 4y = -9.$

13. $ax + by = a - b,$
$ax - by = a + b.$

14. $x + my = -m^2,$
$x + ny = -n^2.$

15. $ax + by = 2ab,$
$bx + ay = a^2 + b^2.$

Solve, by any method:

16. $3x + 4y = 7,$
$4x + y = 3.$

17. $x + 2y = 8,$
$3x + y = 9.$

18. $8x - y = 34,$
$x + 8y = 53.$

19. $3x = 4y,$
$5x = 6y - 4.$

20. $3x + 4y + 5 = 0,$
$6x + 7y + 8 = 0.$

21. $ax + by = 1,$
$cx + dy = 1.$

22. $cx + dy = a,$
$mx + ny = b.$

23. $2(x - 2) = 3(y + 3),$
$3(x - 2) + 2 = 5(y + 3).$

24. $(r - 1)(s + 2) = (r - 3)(s - 1) + 8,$
$4(2r - 1) - 15(s - 2) = 20.$

25. $(b - a)x + (a + b)y = a^2 + b^2,$
$bx - 2ay = b^2 - 2a^2.$

26. $m - n = b - a,$
$am - ac = bn - bc.$

27. $(m + n)x + (m + p)y = m + n,$
$(m + p)x + (m + n)y = m + p.$

91. The method of solution by determinants may be extended to systems containing three or more equations.

For example: $ax + by + cz = d,$ (1)

$ex + fy + gz = h,$ (2)

$kx + ly + mz = n.$ (3)

Using the addition and subtraction method, we obtain after some prolonged effort,

$$x = \frac{dfm + hlc + ngb - cfn - gld - mhb}{afm + elc + kgb - cfk - gla - meb}, \tag{4}$$

$$y = \frac{ahm + enc + kgd - chk - gna - med}{afm + elc + kgb - cfk - gla - meb}, \tag{5}$$

$$z = \frac{afn + eld + khb - djk - hla - neb}{afm + elc + kgb - cfk - gla - meb}. \tag{6}$$

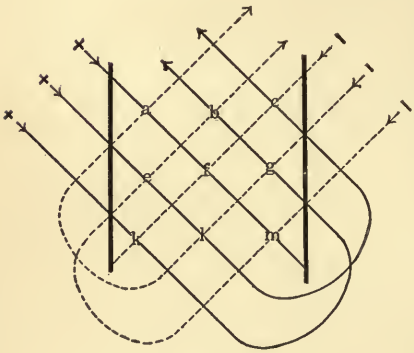

FIG. 7.

These numerators and denominators may be written down more conveniently in the form of determinants. The six terms of the denominator may be written

$$\begin{vmatrix} a & b & c \\ e & f & g \\ k & l & m \end{vmatrix},$$

provided that the following rule of expansion is observed (Fig. 7):

(1) Take the positive of the products of the terms marked by the + arrows.

The three products are $+ afm$, $+ elc$, $+ kgb$.

(2) Take the negative of the products of the terms marked by the − arrows.

The three products, with their signs changed, are $- cfk$, $- gla$, $- meb$.

It follows then that

$$\begin{vmatrix} a & b & c \\ e & f & g \\ k & l & m \end{vmatrix} = afm + elc + kgb - cfk - gla - meb.$$

This process of expansion must be fixed thoroughly in the mind. The numerators of (4), (5), and (6) can be expressed by the determinants in the same way.

Three independent linear equations in three unknown quantities can be solved by determinants according to the following rules:

I. *In the three denominators of the values of x, y, and z, the determinant is the same. It is formed by writing the coefficients of x, y, z, as they stand in equations* (1), (2), (3).

II. *The determinant representing the numerator of* (4) *is formed from the denominator by replacing the column containing a, e, k, the coefficients of x, by the absolute terms d, h, n.*

III. *The determinant representing the numerator of* (5) *is formed from the denominator by replacing the column containing b, f, l, the coefficients of y, by the absolute terms d, h, n.*

IV. *The determinant representing the numerator of* (6) *is formed from the denominator by replacing the column containing c, g, m, the coefficients of z, by the absolute terms d, h, n.*

Example. Solve:

$$x + y + z = 6,$$
$$2x - y + z = 5,$$
$$3x - 4y + 2z = 3.$$

The denominator is $\begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 3 & -4 & 2 \end{vmatrix}$

$$=1\times(-1)\times2+2(-4)\times1+3\times1\times1-1\times(-1)\times3-1\times(-4)\times1$$
$$-2\times2\times1=-4$$

The numerator for x is $\begin{vmatrix} 6 & 1 & 1 \\ 5 & -1 & 1 \\ 3 & -4 & 2 \end{vmatrix}$

$$=6\times(-1)\times2+5\times(-4)\times1+3\times1\times1-1\times(-1)\times3-1\times(-4)\times6$$
$$-2\times5\times1=-12.$$

The numerator for y is $\begin{vmatrix} 1 & 6 & 1 \\ 2 & 5 & 1 \\ 3 & 3 & 2 \end{vmatrix}$

$$=1\times5\times2+2\times3\times1+3\times1\times6-1\times5\times3-1\times3\times1-2\times2\times6=-8.$$

The numerator of z is $\begin{vmatrix} 1 & 1 & 6 \\ 2 & -1 & 5 \\ 3 & -4 & 3 \end{vmatrix}$

$$=1\times(-1)\times3+2\times(-4)\times6+3\times5\times1-6\times(-1)\times3-5\times(-4)\times1$$
$$-3\times2\times1=-4.$$

Hence
$$x=(-12)\div(-4)=3.$$
$$y=(-8)\div(-4)=2.$$
$$z=(-4)\div(-4)=1.$$

EXERCISES

92. Solve:

1. $x+y+z=9,$
   $x-y+z=3,$
   $x+y-z=1.$

2. $2x-3y+z=-1,$
   $3x+2y-z=4,$
   $4x-y+5z=17.$

3. $a+2b-c=1,$
   $8a+4b+3c=29,$
   $3a-5b-4c=-11.$

4. $m+n=3,$
   $p+m=12,$
   $n+p=5.$

5. $r+2s=10,$
   $3r+4t=12,$
   $5s+6t=22.$

6. $x+y+z=2,$
   $3x-4y+5z=41,$
   $9x-13y=-57.$

7. $x-y-z=-\frac{1}{12},$
   $2x-3y+4z=1,$
   $4x+6y+z=4\frac{1}{4}.$

8. $x+y+z=a,$
   $x-y+z=b,$
   $x+y-z=c.$

9. $ax+by=m,$
   $cy+dz=n,$
   $ex+fz=p.$

10. $r+s+t+u=0,$
    $2r+3s+4t+5u=-6,$
    $3r-4s+5t-6u=2,$
    $4r+5s-2t-u=2.$

PROBLEMS

93. 1. A man has 20 coins, all dollars and quarters, amounting to \$13.25. How many coins of each denomination has he?

2. Weights of 13 and 17 lb. are fastened to the ends of a 9-foot pole. Where must the pole be supported in order that the weights will balance?

3. Two men carry a weight of 300 lb. by means of a pole from which the weight is suspended. One man holds the pole at a distance of 3.7 feet from the weight, the other at a distance of 4.3 feet. How much of the weight does each man carry?

4. Two unknown weights balance when suspended 11 and 12 inches respectively from the fulcrum. If the weights are interchanged, 23 ounces must be added to the lesser weight to restore the balance. Find the two weights.

5. A farm laborer received $2.75 and board for every day he worked, and paid $.75 for board every day he was idle. After 60 days he received $154.50. How many days was he idle?

6. A man has $12,000 at interest in two investments. From one he receives 5% interest, from the other 6%. His income from these investments is $645 per year. How is the money divided?

7. Find the capital and the receipts of a railway company, if its receipts are distributed as follows:

(1) Payment of a guaranteed dividend of 4% on $\frac{1}{4}$ of the capital stock.

(2) Payment of a dividend of 3% on the remaining capital stock, this dividend amounting to $162,000.

(3) To working expenses and a reserve fund, which absorb 55% of the entire receipts.

8. The area of a trapezoid is the product of half the sum of the two bases and the altitude. Find the two bases, if the altitude is 12 in., the difference between the bases is 3 in., and the area of the trapezoid is 150 sq. in.

9. How many gallons of each of two liquids, one 25% alcohol, and the other 75% alcohol, must be taken for a 17-gallon mixture that is 45% alcohol?

10. How many ounces of silver, 75 % pure and 85 % pure, must be mixed to give 20 ounces of silver, 83 % pure?

11. Tin and lead occur in the following ratios: In plumber's solder 1 : 2; in soft solder 2 : 1; in common pewter 4 : 1. How many pounds of common pewter and plumber's solder must be taken to make 10 lb. of soft solder?

DIVISION BY ZERO IMPOSSIBLE

94. To avoid perplexing errors in algebraic transformations it should be made very clear that it is not permissible to divide by zero.

Let a and b be two fixed numbers. Both a and b are finite numbers, because in elementary algebra there are no fixed numbers that are infinite.

Letting $\frac{a}{b} = x$, we *define* the quotient $\frac{a}{b}$ as the value of x in the equation $bx = a$.

I. As long as a and b are fixed numbers different from zero, it is always possible to find a value for x.

II. If a is not zero, but b is zero, we are asked to find a number x which, multiplied by zero, gives a as a product.

But every fixed number, when multiplied by zero, gives the product zero.

Hence no number x exists which satisfies the equation $0 \cdot x = a$.

If, therefore, we perform operations which amount to division by zero, we must not be surprised if we obtain absurd results.

Point out the error in the following:

Let	$x = c.$
Multiply both sides of $x = c$ by x,	$x^2 = cx.$
Subtract c^2 from both sides,	$x^2 - c^2 = cx - c^2.$
Factor,	$(x + c)(x - c) = c(x - c).$
Divide by $(x - c)$,	$x + c = c.$
Since $x = c$,	$2c = c.$
Divide both sides by c,	$2 = 1.$

EQUATIONS WITH FRACTIONS

95. When an equation involves fractions we usually begin its solution by clearing it of fractions. This operation rests on the principle that when equals are multiplied by equals, the results are equal.

1. Solve $$\frac{x}{m+n} - n = \frac{m}{m-n}.$$

Observe that this equation is meaningless and absurd when $m = n$, for in that case it involves a division by zero. For the same reason we cannot have $m = -n$, for then $m + n = 0$. Let us assume that neither denominator is zero.

Then the l. c. d. $= (m+n)(m-n)$. Multiply both sides by $(m+n)(m-n)$.

We obtain, $$(m-n)x - n(m^2 - n^2) = m(m+n).$$

Transpose, $$(m-n)x = n(m^2 - n^2) + m(m+n).$$

Divide both sides by $(m-n)$, $$x = \frac{n(m^2 - n^2) + m(m+n)}{m-n}.$$

2. Solve $$\frac{x}{x-1} + \frac{x-3}{x^2-1} = \frac{5}{4}. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Observe that if $x = 1$, or $x^2 = 1$, this equation represents an absurdity, since it involves division by zero.

On the supposition that $x^2 \neq 1$, the l. c. d. $= 4(x^2 - 1)$.

Multiply both sides by $4(x^2-1)$, $$4x(x+1) + 4(x-3) = 5(x^2-1). \quad (2)$$

Simplify, $$4x^2 + 4x + 4x - 12 = 5x^2 - 5.$$

Transpose, $$x^2 - 8x + 7 = 0.$$

Factor, $$(x-7)(x-1) = 0.$$

Hence, $$x = 7,\ x = 1.$$

Substitute in the original equation (1); we find $x = 7$ satisfies it, but $x = 1$ does not satisfy it, since it gives rise to division by zero. We must reject $x = 1$; $x = 7$ is the only correct solution of (1).

It is of interest to notice that, while $x = 1$ does not satisfy (1), it does satisfy (2). In other words, equation (2) is satisfied by two values of x, while equation (1) is satisfied by only one value of x.

It is clear that the new root was introduced in the act of multiplying both sides of (1) by $4(x^2 - 1)$. Such new roots

which do not satisfy the original equation, but do satisfy an equation that is derived from it, are called *extraneous roots.*

When we multiply both sides of an equation by an expression involving x, we obtain a new equation some of whose roots may not satisfy the original equation and are not correct solutions of it; they are extraneous roots. This may happen even though all operations are performed free of error.

Hence, to make sure that our values of x are correct, we must *substitute the values found for x in the original equation,* to see whether it is satisfied or not.

EXERCISES

96. Solve and check:

1. $\dfrac{2}{x-1}=\dfrac{1}{x-3}.$

2. $\dfrac{5}{y}+4=\dfrac{4}{y}+5.$

3. $\dfrac{3}{2r+1}=\dfrac{1}{r-1}.$

4. $\dfrac{4}{7s+3}=\dfrac{3}{6s+2}.$

5. $\dfrac{x+1}{x+2}=\dfrac{x+3}{x+5}.$

6. $\dfrac{3m-1}{3m+4}=\dfrac{3m-5}{3m+2}.$

7. $\dfrac{1}{(x-1)(x-4)}=\dfrac{1}{(x-2)(x-5)}.$

8. $4-\dfrac{5}{y+1}=7-\dfrac{3y-8}{y-3}.$

9. $(a^2-1)x=a+1.$

10. $(m^2-4)y=m-2.$

11. $\dfrac{x}{a}=\dfrac{1}{c}.$

12. $\dfrac{cx}{b}=\dfrac{1}{d}.$

13. $\dfrac{x}{b}-1=a.$

14. $\dfrac{x}{m+n}+m=n.$

15. $\dfrac{cx}{a}+\dfrac{ax}{c}=3.$

16. $\dfrac{3}{4x+3}=\dfrac{7}{6x-1}.$

17. $\dfrac{1}{x-2}+\dfrac{2}{x-3}=\dfrac{3}{x-4}.$

18. $\frac{5}{x-5}-\frac{4}{x-4}=\frac{1}{x-3}.$

19. $\frac{1}{x-1}-\frac{1}{x-3}=\frac{1}{x-5}-\frac{1}{x-7}.$

20. $\frac{3}{r+3}-\frac{1}{3-r}=\frac{5r+15}{r^2-9}.$

Solve and tell, by substitution in the original equation, which of the roots, if any, are extraneous.

21. $\frac{1}{x-1}+\frac{2}{x-2}=\frac{5}{2}.$

22. $\frac{6}{x+6}-\frac{5}{x-7}=\frac{3(x-20)}{(x+6)(x-7)}.$

23. $\frac{5x^3-5}{5x^2-9x+5}=x+3.$

24. $\frac{x^2+6x+9}{x^2-9}=2x-6.$

25. $\frac{3y-1}{y}=\frac{2y}{2y-1}.$

26. $\frac{2x}{x+1}-\frac{8}{2x^2-x-3}=\frac{-3}{2x-3}.$

27. $\frac{5}{2x-3}+\frac{3}{3x+4}=\frac{-x^2+34x-31}{6x^2-x-12}.$

28. $\frac{1}{y-b}-b=\frac{2b}{y^2-b^2}.$

MISCELLANEOUS PRACTICAL PROBLEMS

97. Applications of algebra, such as occur in the problems which follow, are frequently made by builders and manufacturers. Certain formulas which have been established by engineers and mathematicians are used in computing the desired results. We shall assume the formulas as true without giving the mode of deriving them. Care must be taken to use in every problem the proper units of measure.

1. Find correctly, to two decimal places, the side of a square whose area is 35.06 sq. in.

2. Compute the altitude of an equilateral triangle whose sides are a inches.

3. If one side of a rectangle is a inches long, and one of the adjacent sides is three times as long, find the length of the diagonal.

4. A ladder of the length $9\,a$ is placed against a wall, with its foot at a distance $4\,a$ from the wall. How high above the ground is the top of the ladder?

5. In placing blackboards in schoolrooms, architects determine the height of the chalk rail above the floor by the formula

$$h = 25 + \tfrac{3}{2}(g - 4),$$

where $h =$ height of the chalk rail in inches, $g =$ the number of the grade to which the pupils in the room belong. Find the height of the chalk rail for pupils in the 5th grade; also for pupils in the 8th grade and in the 3d grade.

6. The Baldwin Locomotive Works in Philadelphia have derived a formula, $R = 3 + \frac{v}{6}$, for finding the approximate relation between the resistance, R, offered by a railway train, and the velocity, the train traveling on a straight and level track. In this formula R is the resistance in pounds per ton weight of the train, v is the velocity of the train in miles per hour. What is the resistance *per ton* of a train running at the

rate of 40 mi. an hour? What is the resistance of this train, if it weighs 100,000 T.?

7. The load that may be safely applied to an iron chain is given by the formula, $l = 7.11\, d^2$, where l is the load in tons (2000 lb.), and d is the diameter of the iron chain in inches. Find the largest safe load that may be applied to a chain for which d is $\frac{3}{4}$ in. Find d when $l = 13$ T.

8. The distance which one can see from an elevation at the sea has been found to be $d = 1.23\sqrt{h}$, where d is the number of miles one can see, and h is the elevation of the observer in feet. How far is the horizon from a man standing at the seashore, whose eye is 6 ft. from the ground?

9. In Ex. 8, how far can one see from a cliff 27 ft. high? 67 ft. high? How high must a cliff be to afford a view of 20 mi.?

10. From a cliff I can just see the lights of a seaport 12 mi. across the sea. If these lights are 20 ft. above the sea, what is the height of my eye?

11. According to a rule sometimes used by architects, the "rise" (r inches) in the steps of a staircase (Fig. 8) is connected with the "tread" (t inches), by the formula $r = \frac{1}{2}(24 - t)$, where r and t are also subject to the limitations, $7 > r > 5\frac{1}{2}$, $12 > t > 10$. Explain the formula in words. What is the "rise" when the tread is $10\frac{1}{2}''$? $11\frac{3}{4}''$?

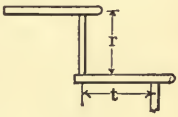

FIG. 8.

12. Engineers have determined that the velocity (v feet per minute) of a stream at the bottom of a river is connected with the velocity (V feet per minute) at the surface, by the formula $v = V + 1 - 2\sqrt{V}$. Find the velocity at the bottom when the velocity at the surface is 5 ft. per minute, 8 ft. per minute, 95 ft. per minute.

13. A rectangular beam (Fig. 9) is supported at the ends and loaded in the middle. The weight it will just bear, without breaking, is $w = \frac{kbd^2}{l}$, where w is the weight in pounds, b the breadth of the beam in inches, d is the depth of the beam in inches, l the length of the beam in feet, and k a coefficient equal to 3470 for wrought iron, 2540 for cast iron, 450 for red pine. Find the greatest weight which can be hung in the middle of a wrought-iron bar 5 ft. long, 1 in. wide, and 2 in. deep.

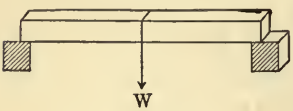

FIG. 9.

14. In Ex. 13, find w when the beam is cast iron, also when it is red pine.

15. In Ex. 13, how deep must a beam of red pine be, to support 2 tons, if it is 6 ft. long and 3 in. wide ?

16. At a curve in a railway track the outer rail is raised above the inner rail by an amount indicated by the formula $h = \frac{wv^2}{1.25\,r}$, where h is the number of inches that the outer rail is elevated above the inner, w is the width between the rails in feet, v is the greatest speed of a train in miles per hour, r is the radius of the curve in feet. If $r = 2000$ ft., $w = 56\frac{1}{2}$ in., $v = 60$ mi. per hour, what is h?

17. The safe load which can be applied to a rope is $l = k_1c^2$, its breaking load is $b = k_2c^2$, where l and b are measured in tons, and c (the circumference of the rope) in inches. For common hemp ropes $k_1 = .032$, $k_2 = .18$; for best hemp rope, $k_1 = .100$, $k_2 = .60$; for steel-wire rope, $k_1 = .450$, $k_2 = 2.8$. Calculate the safe load (*a*) for a common hemp rope 3.5 in. in circumference, (*b*) for a steel-wire rope 2.5 in. in circumference, (*c*) for a best hemp rope 4 in. in circumference. Calculate the breaking load for each of these.

18. Draw a graph showing the safe load of steel-wire ropes for different circumferences, up to 4 inches.

19. The breaking load of the best hemp rope is given by the formula, $b = .60\,c^2$, where b is the breaking load in tons, c is the circumference of the rope in inches. Draw a graph showing the breaking load from $c = .3$ in. to $c = 4$ in. From the graph determine the breaking load for three values of c not used in constructing the graph.

20. In the flow of water through pipes, a certain head is necessary merely to overcome the friction of the water against the pipes. This head is given by the expression, $h = \frac{l}{d}\left(.0036 + \frac{.0043}{\sqrt{s}}\right)\frac{s^2}{32}$, where s is the speed of the water in feet per second, l is the length of the pipe in feet, d is the diameter of the pipe in inches, h is the head of the water in feet. Find the head required to overcome the friction in a 3-inch pipe, 3000 feet long, in which water is running at the rate of 3 feet per second.

21. A house is to rest on piles as a foundation. In the erection of it one must know how much each pile can support. This is computed from the performance of the "pile driver," a machine that lifts a heavy "ram" and drops it on the pile. The maximum load l, in tons, that a pile will bear is obtained from the formula, $l = \frac{wh}{d(w+p)}$, where w is the weight of the ram in cwt., h is the height in feet from which the ram falls, d is the distance in inches that the pile was driven in by the last blow, p is the weight of the pile in cwt. If $w = 6$, $h = 4$, $p = 15$, $d = 1\frac{1}{2}$, what is the maximum load the pile will safely bear?

22. A steam engine is operated by steam which enters a cylinder C (Fig. 10) and presses against one side of the piston P and moves it; the motion is transmitted by the piston rod R to the crank shaft R_1, which turns the wheel around. The rate

at which the engine does mechanical work depends upon the size of the cylinder, the pressure of the steam, and the number of revolutions per minute. The rate of doing work is measured in horse powers, H. If d is the diameter of the cylinder in

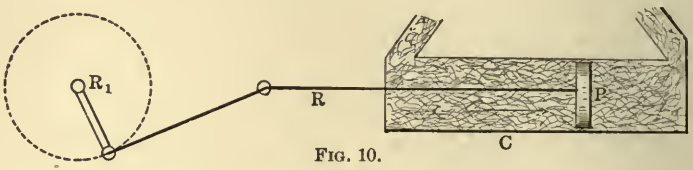

FIG. 10.

inches, l the length of the piston stroke in feet, n the number of revolutions of the wheel per minute, and p the mean steam pressure per square inch, then $H = \frac{nlpd^2}{21000}$. If $n = 22$, $d = 55''$, $l = 51''$, $p = 20$ lb., find H. If $n = 25$, $p = 15$ lb., $l = 4\frac{1}{2}$ ft., what must d be, that the horse power may be 30?

23. The horse power required to move a ship is given approximately by the expression $H = .0088\, s^3\, (.05\, A + .005\, S)$, where s is the speed in knots, A is the immersed cross section of the ship in square feet, S is the wetted surface of the ship in square feet. A and S remaining the same, state the nature of the variation of H as a function of s. What horse power is needed to move a steamer at 12 knots, when $A = 230$ and $S = 3100$?

CHAPTER III

PROPORTION, VARIATION, FUNCTION

PROPORTION

98. A *proportion* expresses the equality of two fractions. The fractions are sometimes called *ratios*.

Thus, $\frac{a}{b}=\frac{c}{d}$ is a proportion. It is written also $a:b=c:d$.

In $a/b=c/d$, the terms a and d are called the *extremes*, the terms b and c the *means*. In the ratio a/b, a is sometimes called the *antecedent*, b the *consequent*, but ordinarily there is no need of these terms.

A *mean proportional* between two numbers a and b is the number m which satisfies the equation $\frac{a}{m}=\frac{m}{b}$.

Since a proportion is really an equation, it is treated like an equation.

Thus, in the proportion, $$\frac{a}{b}=\frac{c}{d}.$$

1. Multiply both sides by bd, $$ad=bc.$$

This relation is expressed by the theorem:

In a proportion, the product of the means is equal to the product of the extremes.

2. When only three terms of a proportion are known numbers, the fourth term may be found.

Thus, if a, b, and c are known, we obtain from $\frac{a}{b}=\frac{c}{d}$,

$$ad=bc$$

and $$d=\frac{bc}{a}.$$

3. Add 1 to both sides of the equation, $$\frac{a}{b}+1=\frac{c}{d}+1,$$

Whence, the new proportion, $$\frac{a+b}{b}=\frac{c+d}{d}.$$

EXERCISES

99. 1. Prove that if the product of two numbers, ef, is equal to the product of two other numbers, gh, one pair may be taken as the means, the other pair as the extremes, of a proportion.

Prove that, if $\frac{a}{b}=\frac{c}{d}$, then

2. $\frac{a-b}{b}=\frac{c-d}{d}$.

3. $\frac{a-b}{a+b}=\frac{c-d}{c+d}$.

4. $\frac{a}{a+b}=\frac{c}{c+d}$.

5. $\frac{a}{a-b}=\frac{c}{c-d}$.

6. $\frac{a+b}{a-b}=\frac{c+d}{c-d}$.

7. $\frac{a}{c}=\frac{b}{d}$.

8. $\frac{a^2}{b^2}=\frac{c^2}{d^2}$.

9. $\frac{a^3}{b^3}=\frac{c^3}{d^3}$.

10. $\frac{a^n}{b^n}=\frac{c^n}{d^n}$.

11. $\frac{a^4}{a^2b^2}=\frac{c^2d^2}{d^4}$.

Simplify the following ratios by writing each in the form of a fraction and reducing the fraction to its lowest terms:

12. $125:675$.

13. $69\,a^3c^2:46\,a^2bc$.

14. $a^2-b^2:(a+b)c$.

15. $(a-b)^2:a^2-b^2$.

16. $(8\,m^3+n^3):16(2\,m+n)$.

17. $4(p^3-8\,q^3):(p-2\,q)$.

18. $(a^4+a^2b^2+b^4):(a^2-ab+b^2)$.

19. $(x^2-xy-2\,y^2):3(x-2\,y)$.

20. $\left(\frac{1}{9}a^2-\frac{1}{4}\frac{b^2}{c^2}\right):(2\,ac+3\,b)$.

21. $\left(\frac{1}{x^3}-\frac{27}{y^3}\right):\left(\frac{1}{3\,x}-\frac{1}{y}\right)$.

22. $(x^6-2\,x^3y^3+y^6):(x^2+xy+y^2)$.

23. $(x^4+1):(x^2-x\sqrt{2}+1)$.

24. Find the mean proportional between 31.2 and 0.96.

25. Find the mean proportional between $3\,a^2c^2$ and $7\,c^2d^3$.

26. Determine which is the greater of the ratios 19 : 25, or 56 : 74.

27. Two numbers are in the ratio of 3 : 4, and their difference is to their product as 1 is to 18. Find the numbers.

28. Find the last term of a proportion whose first three terms are 27, 6, and 18.

29. Divide \$ 644 among A, B, and C, so that A's share is to B's in the ratio of 2 to 3, and B's share is to C's in the ratio of 5 to 7.

30. The income of an estate should be divided between A and B in the ratio of 5 : 4; \$ 1100, however, is paid to A, and \$ 700 to B. Which has been underpaid, and by how much?

FUNCTIONS

100. The formula $$s = 16.1\,t^2$$

is used in finding the distance (space) through which a body falls from rest, in time t, where the time is measured in seconds and the distance in feet.

During the fall, the time t and the distance s both increase; t and s are *variables*, while the number 16.1, of course, is *constant.*

For every value of t it is possible to compute s, s being dependent for its value upon the value of t; s is said to be a *function* of t.

A variable is a function of another variable when its value is determined for every value of the other variable.

We have used variables x and y in the drawing of graphs. In the equation of the straight line, $y = 3\,x + 1$, x and y are variables; 3 and 1, of course, are constants; y is a *function* of x, since for every value of x there exists a definite value of y.

ORAL EXERCISES

101. 1. The velocity of a body falling from rest is given by the formula,

$$v = 32.2\,t,$$

where v is the velocity in feet per second and t is the time of fall in seconds.

Which are the variables? Which is the constant? Why is v a function of t?

2. The area of a circle is given by the formula

$$A = \pi r^2.$$

Why is A a function of r? Is π a constant or a variable?

3. If a man's monthly salary is \$125, his earnings may be expressed by the equation,

$$E = 125\,n,$$

where n is the number of months during which he draws his salary, and E the earnings expressed in dollars.

In this formula explain "constant," "variable," and "function."

4. The area of a rectangle is given by the formula

$$A = bh,$$

where b and h are the "base" and "height" in linear units, and A is the "area" in the corresponding square units. Here A varies when b varies and also when h varies. That is, *A is a function of the two variables, b and h.* Find A, when $b = 2$, $h = 3$; also when $b = 4$, $h = 2$.

DIFFERENT MODES OF VARIATION

102. When one variable is a function of another variable, the variation takes on different forms in different cases. In working problems it is necessary to know the exact mode of variation, else absurd results will be obtained.

In geometry it is proved that the length of a circle varies as its radius, that the area of a circle varies as the *square* of its radius, that the volume of a sphere varies as the *cube* of its radius.

In symbols,

$$C : C_1 = r : r_1,$$
$$A : A_1 = r^2 : r_1^2,$$
$$V : V_1 = r^3 : r_1^3.$$

Hence, if the second radius is twice the first ($r_1 = 2r$), then the length of the second circle is twice that of the first, the area of the second circle is four times that of the first, the volume of the second sphere is eight times that of the first.

Of special interest and importance are the problems in which one variable varies *directly* as another, or *inversely* as another. The words "directly" and "inversely" have the same significance in algebra as in arithmetic.

For example, at any moment, the *length* (L) of the shadow of a vertical post cast upon level ground depends upon the *height* (H) of the post; the taller the post, the longer the shadow. Here L varies directly as H; we write,

$$L : L_1 = H : H_1, \text{ or } L = kH, \text{ or } L \propto H,$$

where k is a constant and where $L \propto H$ signifies "L varies as H." These three notations denote the same relation between L and H; all three indicate that L is a function of H.

The time (t) it takes a train to travel from New York to Chicago depends upon its speed (v); the *greater* the speed, the *less* the time. Here t varies *inversely* as v; we write

$$t : t_1 = v_1 : v, \text{ or } t = k \cdot \frac{1}{v}, \text{ or } t \propto \frac{1}{v}.$$

From $t = k \cdot \frac{1}{v}$ we obtain $vt = k$; that is, if one variable varies inversely as another, the product of the two variables is a constant.

Sometimes there are three variables. When one variable depends upon two other variables in such a way as to *vary as the product of the two variables*, we say that the first variable varies *jointly* as the two other variables.

Ex. 4 in § 101 was of this sort. If we double the length of the base (b) of a triangle and at the same time double its height (h), the area (A) is four times greater. In symbols, the variation is expressed thus,

$$A : A_1 = bh : b_1h_1, \text{ or } A = k \cdot bh, \text{ or } A \propto bh.$$

There are many other kinds of variation. A variable may vary *directly* as a second variable and *inversely* as a third, as indicated by the formula $y = k\frac{x}{z}$. Or, a variable may depend upon three or more other variables. Thus, the price of lace depends not only upon the number of yards, but also upon the quality of the article, the cost of labor, the rate of profit exacted by the seller. Practical problems involving variables as functions of other variables and exhibiting many different forms of variation were given in § 97.

VARIATION SHOWN IN GRAPHS

103. Consider a proportion in which two of the terms are variables.

An automobile travels 29 miles in 2 hours; at this rate, how far does it travel in h hours?

As the distances are proportional to the times of travel, we have the proportion,

$$d : 29 = h : 2,$$

where d indicates the number of miles and h the number of hours. Here d and h are variables; when h increases, d increases also, the value of d being dependent upon that of h.

The relation between h and d can be expressed equally well by changing the equation to the form

$$d = \frac{29}{2}h.$$

This shows that h and d vary in such manner that the number d is always $\frac{29}{2}$ as large as the number h; d is a *function* of h.

This *variation* is exhibited by the following graph (Fig. 11). Hours are measured along the horizontal line OX; miles are measured along the vertical line OY. For convenience the divisions representing hours are taken ten times longer than the divisions representing miles. The line OA is constructed as follows:

When $h = 0$, then $d=0$. This determines the point O.

When $h = 4$, then $d = 58$. This determines the point A.

Through O and A draw the line OA.

Using this graph we can tell *by inspection* the distance traveled during a time not exceeding 5 hours. For instance, to tell the distance traveled in 3.2 hr., we find on OX the point C, then proceed parallel to OY to the point B; and from B to the left, parallel to OX. The answer is 46^+ miles. Actual multiplication gives 46.4 miles.

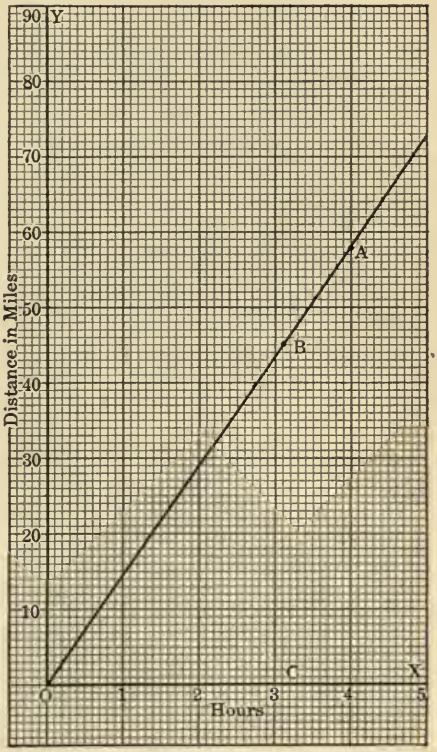

FIG. 11.

How accurately can the distance be measured by this graph?

How accurately can the time be measured by this graph?

How can a graph be drawn that will give distances correctly to $\frac{1}{10}$ of a mile?

EXERCISE

104. 1. From the graph, Fig. 11, tell the distance traveled in 2.5 hr., 3.4 hr., 4.1 hr., 4.9 hr.

2. From the same graph tell the time needed for traveling 24 mi., 50 mi., 35 mi., 65 mi., 48 mi.

3. Draw a graph that will show the railroad fare for traveling short distances at the rate of $2\frac{2}{3}$ cents a mile.

4. Draw a graph showing corresponding values of x and y in the proportion $x : y = 1.9 : 2.3$.

5. A clerk in a New York store converts prices in "marks-per-meter" into "dollars-per-yard." He draws a graph and roughly "checks" his computations by it. If 1 m. = 1.1 yd., and 1 M. = \$ .24, draw the graph.

Hint. 1 M. per 1 m. = \$ .24 per 1.1 yd. = \$? per 1 yd.

6. One knot (nautical mile) is equal to 1.15 miles. Draw a graph for changing from one unit to the other and then tell from it the speeds, in miles per hour, of cruisers making 21, 22, 24, 27, 29 knots per hour.

7. In lifting weights with the aid of a small, well-lubricated screw jack, part of the power varies with the weights that are raised, and the rest of the power is a fixed amount to overcome friction. It was found by trial that 5 lb. was necessary to lift a weight of 100 lb., and that 2.8 lb. was necessary to lift 50 lb. Derive an equation and draw a graph showing the relation between the power P and the weight W.

By the conditions of the problem, the relation between P and W can be expressed by a linear equation of the form $P = aW + b$, where P and W are the variables, and a and b are constants to be found from our data. Observe that aW is the part of the power which varies with the weight, and that b is the constant part that is expended to overcome friction. Notice that P is a function of W.

When $P = 5$ lb., $W = 100$ lb., hence $5 = 100\,a + b$.

When $P = 2.8$ lb., $W = 50$ lb., hence $2.8 = 50\,a + b$.

Solving for a and b we obtain $a = .044,\ b = .6.$

Substituting these values we obtain $P = .044\ W + .6,$

which expresses "the law of the machine." The graph of this equation is shown in Fig. 12.

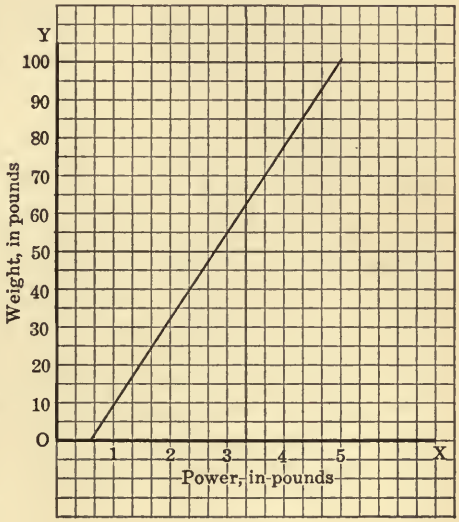

FIG. 12.

8. In Fig. 12, tell by inspection the power, P, necessary to raise a weight, W, of 30 lb., of 40 lb., of 65 lb., of 75 lb. To what fraction of a pound can you estimate the power by this graph?

9. In Fig. 12, tell by inspection the weight W, which can be raised when P is $2\frac{1}{3}$ lb., 3 lb., $4\frac{2}{3}$ lb. By this graph, can W be determined as closely as 1 lb.? How would you draw a graph yielding more accurate results?

10. In a system of well-lubricated pulleys a weight of 50 lb. is raised by a power of 10.7 lb., and a weight of 80 lb. is raised by a power of 16.7 lb. If part of the power varies as the weight, and part of the power is fixed, find the equation or "law of the machine" which shows the relation between P and W. Draw a graph exhibiting simultaneous values of P and W.

PROBLEMS

105. 1. If $x \propto y$, and $x = 5$ when $y = 6$, find x when $y = 7$.

First Solution. Since $x \propto y$, the variation is direct.

Hence, $y : y_1 = x : x_1.$

Taking $y_1 = 6$, $x_1 = 5$, $y = 7$, $7 : 6 = x : 5.$

Hence, $x = 5\frac{5}{6}.$

Second Solution. Since $x \propto y,$

we have $x = ky.$

To determine the constant k, use the given simultaneous values of the variables x and y, viz., $x = 5$ when $y = 6$. We obtain

$$5 = k \cdot 6.$$

Hence, $k = \frac{5}{6}.$

Substituting the value of k, $x = \frac{5}{6}y.$

When $y = 7$, we have $x = \frac{5}{6} \times 7 = 5\frac{5}{6}.$

The student should master both solutions.

2. If x varies *inversely* as y, and $x = 5$ when $y = 6$, find x when $y = 7$.

First Solution. Since $x \propto \frac{1}{y}$, we obtain the proportion

$$y : y_1 = x_1 : x,$$
$$7 : 6 : 5 : x.$$
$$x = 4\frac{2}{7}.$$

Second Solution. Since $x \propto \frac{1}{y}$,

$$x = \frac{k}{y}.$$

Taking $x = 5$, $y = 6$, $5 = \frac{k}{6},$

$$k = 30.$$

Substituting the value of k, $x = \frac{30}{y}.$

When $y = 7$, $x = \frac{30}{7} = 4\frac{2}{7}.$

3. If z varies *jointly* as x and y, and $z = 5$, when $x = 3$ and $y = 4$, find z when $x = 7$ and $y = 6$.

First Solution. By proportion, $z : z_1 = xy : x_1y_1$.

Take $x_1 = 3,\ y_1 = 4,\ z_1 = 5,\ x = 7,\ y = 6.$

Then, $z : 5 = 42 : 12,$

$$z = 17\tfrac{1}{2}.$$

Second Solution. Since $z \propto xy$,

$$z = kxy.$$

To determine k, substitute the simultaneous values,

$$z = 5,\ x = 3,\ y = 4,$$
$$5 = k \times 3 \times 4,$$
$$k = \tfrac{5}{12}.$$

Hence, $z = \frac{5}{12}xy.$

When $x = 7,\ y = 6,$ $z = \frac{5}{12} \times 42 = 17\frac{1}{2}.$

4. If z varies directly as x and inversely as y, and $z = 5$, when $x = 3$, $y = 4$, find z when $x = 7$ and $y = 6$.

Solution. Here $z \propto \dfrac{x}{y}$, hence

$$z = k \cdot \frac{x}{y}.$$

Substitute the simultaneous values $z = 5,\ x = 3,\ y = 4,$

$$5 = k \cdot \tfrac{3}{4}.$$

Hence, $k = \frac{20}{3},$

and $$z = \frac{20}{3} \cdot \frac{x}{y}.$$

When $x = 7,\ y = 6,$ $z = \frac{20}{3} \cdot \frac{7}{6} = 7\frac{7}{9}.$

Let the student give the solution by proportion.

5. The distance described by a body falling from rest varies as the *square* of the time. If in 2 seconds it falls 64.4 feet, find the distance it falls in $6\frac{1}{2}$ seconds.

Solution. The distance (d) varies *directly* as the *square* of the time (t). Hence, $d = kt^2.$

When $t = 2$, $d = 64.4$, hence, $64.4 = k \cdot 2^2$,

$k = 16.1$,

and $d = 16.1\, t^2$.

When $t = 6\frac{1}{2}$ seconds, $d = 680.2+$ ft.

6. The brightness or intensity of light varies inversely as the square of the distance from the source of the light. How many times brighter is the page of a book illuminated at a distance of 3 ft. from the source than at a distance of 5 ft.?

Solution. The intensity (i) varies *inversely* as the *square* of the distance (d).

That is, $$i = \frac{k}{d^2}, \text{ or } i : i_1 = d_1^2 : d^2.$$

Since we do not have sufficient data to determine the numerical value of k, it is easier to work the example by proportion.

We obtain $$i : i_1 = 5^2 : 3^2.$$

Or $$\frac{i}{i_1} = \frac{25}{9} = 2\tfrac{7}{9}.$$

At a distance of 3 ft. the page is $2\frac{7}{9}$ times brighter than at 5 ft.

7. If x varies as y, and if x is 144 when y is 3, find the value of y, when x is 360.

8. If x varies as the square of y, and if x is 144 when y is 3, find the value of y when x is 360.

9. If x varies jointly as y and z, and $x = 10$ when $y = 15$, and $z = 18$, find x when $y = 5$ and $z = 24$.

10. If x varies inversely as the cube of y, and $x = 54$ when $y = 3$, find x when $y = .1$.

11. The areas of similar triangles vary as the squares of homologous sides. If homologous sides of two such triangles are 7 and 10, and the area of the larger triangle is 13.5, what is the area of the smaller triangle?

12. The areas of two similar triangles are 35 and 43; one side of the smaller triangle is 5. Find the homologous side of the larger triangle, correct to two decimal places.

13. Two pieces of round timber (right cylinders) have the same altitude; their girths are 3.4 ft. and 3.1 ft. Find the ratio of their volumes.

14. If a carriage wheel 4 ft. 2 in. in diameter makes 240 revolutions in going a certain distance, how many revolutions will a wheel 4 ft. 8 in. in diameter make in going the same distance?

15. The weight of a sphere of given material varies as the cube of the radius. If a sphere having a diameter of 3 in. weighs 4 lb., find the weight of a sphere of the same material, but with a radius of 2 in.

16. The rents of an estate should be divided between A and B in the ratio of 4 : 5. However, A is paid $ 250, and B is paid $ 275. Which has been overpaid, and how much?

17. If a train, whose speed is 55 miles per hour, makes a certain trip in $3\frac{1}{2}$ hr., how long will it take a slower train to travel $\frac{2}{3}$ of that distance, its speed being to that of the fast train, as 4 is to 9?

18. If the pressure p exerted upon the air in a bicycle pump and the volume of the air v obey Boyle's law, according to which $pv =$ constant, and if the pressure is 25 lb. per square inch when the volume is 27 cubic inches, what will the pressure be when the volume is reduced to 20 cubic inches?

19. Part of the expenses of a certain school was fixed; part of them varied with the number of pupils. If the yearly expenses were $ 75,000 when the number of pupils was 710, and $ 85,000 when the number of pupils had increased to 900, find the yearly expense when the number of pupils was 800.

20. If the ratio of $3x + y$ to $4x - 3y$ equals the ratio of 21 to 2, what is the ratio of x to y?

21. At the end of 12 days, 15 men finish one fourth of a piece of work. How many additional men must be engaged in order to complete the remaining work in 18 days?

22. If a body falls from rest through a distance of 16.1 feet during one second, how far will it fall in 8.3 seconds?

23. Compare the intensity of illumination of the page of a book at a distance of 6.3 feet from a source of light with the intensity at a distance of 4.9 feet.

24. If the surface area of a certain sphere is 35 sq. in., what is the surface area of a sphere having a radius 4.7 times longer?

25. If the volume of a certain sphere is 500 cu. in., what is the volume of a sphere whose radius is 2.9 times longer?

26. If the base of a triangle is increased 10-fold and its altitude is reduced to $\frac{1}{3}$ its original length, find the ratio between the initial and final areas.

27. Two casks of similar shape have their homologous linear dimensions in the ratio of 2 : 3.4. Find the ratio of their capacities.

28. The radii of the bases of two cylindrical tanks are 1 ft. 5 in., and 1 ft. 11 in., respectively; the heights of the tanks are 6 ft. and 5 ft., respectively. Compare the capacities of the tanks.

29. The area of a circle is found by the formula $A = \pi R^2$, where $\pi = 3.1416$. Draw a graph by which the areas can be determined by inspection.

R	A	Point
$\frac{1}{2}$	.79	A
1	3.14	B
$1\frac{1}{4}$	4.91	C
$1\frac{1}{2}$	7.07	D
$1\frac{3}{4}$	9.62	E
2	12.56	F
3	28.27	G
4	50.27	H

Assuming values of R, as shown in the table, compute the values of the area A. Explain the construction of Fig. 13.

30. In Fig. 13, find by inspection the areas of circles having radii of $2\frac{1}{4}$, $2\frac{1}{2}$, $2\frac{3}{4}$, $3\frac{1}{4}$, $3\frac{1}{2}$, $3\frac{3}{4}$.

31. If a garrison of 2000 men is provisioned for 40 days, how long will their provisions last if they are joined by 375 other men?

32. The surface of a sphere varies directly as the square of the diameter. If the diameter is 2 inches, the surface is 12.566 square inches. Find the surface when the diameter is 7 inches.

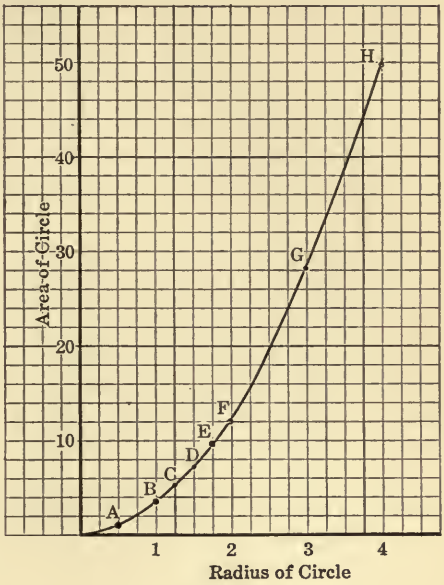

FIG. 13.

33. The value of a piece of land has risen in ten years from \$1200 to \$3200. How would you calculate the rise in value of another similarly situated piece which was worth \$1750 ten years ago?

34. The rent of a house in a town has gone from \$85 to \$70 per month during the last 15 years. If the value of other houses in that neighborhood has decreased in the same ratio, what was the decrease in the rent of a house now rented at \$42?

35. The diagonal of a cube varies directly as the length of an edge. When the edge measures 3 inches, the diagonal measures 5.196 inches. What does the diagonal measure when the edge is 3.8 inches?

36. The weight of a copper coin varies directly as its thickness and also as the square of its diameter. A coin weighing 11 grams has a diameter of 3 cm. and a thickness of .2 cm. What is the weight when the diameter is 2.5 cm. and the thickness .3 cm.?

37. A's rate of working is to B's as 3 : 4; B's rate of working is to C's as 6 : 5. How long will it take C to do work which A can do in 8 hours?

38. The distance through which a body falls from rest varies as the square of the time of falling. If a body falls 257.6 ft. in 4 seconds, how far will it fall in $5\frac{1}{2}$ seconds?

39. If a stone weighing 5 pounds falls from rest through a distance of 144.9 ft. in 3 seconds, how far will a stone fall from rest in 3 seconds, if its weight is 500 pounds? See Ex. 38.

40. The tractive force or pull necessary to move a vehicle at a uniform rate (say 3 miles an hour) varies directly as the pressure (weight of vehicle and load) and inversely as the square root of the average radius of the wheels. If on level, paved, or macadam roads a pull of 57 pounds per ton of pressure is necessary when the front and rear wheels average 50 inches in diameter, what pull is necessary to move 3500 pounds when the wheels of the vehicle average 38 in. in diameter?

41. If a tractive force of 75 lb. is necessary to keep a load of 1 ton in motion in a vehicle whose wheels average 38 in. in diameter, on a dry and hard earth road, what is the tractive force for 5 tons when the wheels average only 26 in. in diameter? See Ex. 40.

42. If on an earth road, in sticky mud $\frac{1}{2}$ in. deep, the tractive force per ton is 119 lb. when wheels 38 in. in diameter are used, what tractive force is required to move 4 tons when the diameters are 50 in.?

43. The tractive force per ton over dry, cloddy plowed ground is 252 lb. for wheels 50 in. in diameter; what is the tractive force for $\frac{1}{4}$ of a ton when the wheels are 26 in. in diameter?

GRAPHS EXHIBITING EMPIRICAL DATA

106. 1. By a certain plan of life insurance a single premium is paid in order that the insured may receive $100 when he is 60 years old or that his beneficiaries may receive $100 if he dies before that age. The premium depends upon the age at which the insurance is taken.

Age next birthday:	15	20	25	30	35	40	45	50
Premium:	$41.20	$45.25	$49.30	$53.80	$59.10	$65.10	$72.05	$80.20

Draw a graph and use it to find the premium one would pay at the age of 18, 23, 32, 38, 43.

It is convenient to mark off along the x-axis the ages *above* 15; along the y-axis the premiums *above* $40. By this device a much smaller sheet of square paper can be used. See Fig. 14.

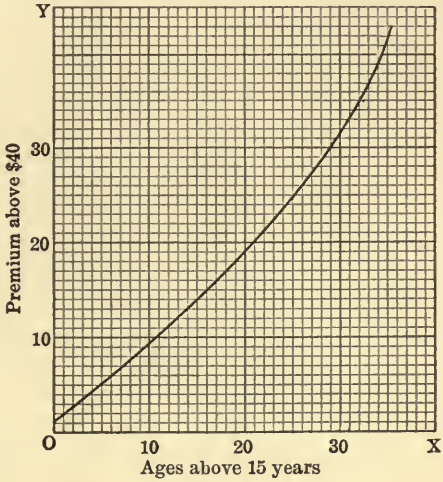

FIG. 14.

2. In a city in the northern part of the United States the times of sunrise on certain days in May, June, and July are as follows:

DAY	MAY	JUNE	JULY
10	4 : 20	3 : 45	3 : 56
20	4 : 04	3 : 44	4 : 07
30	3 : 52	3 : 48	4 : 21

Draw a graph showing the hour of sunrise from May 10 to July 30.

Mark off on one axis the number of days *after* May 10, on the other axis the number of minutes *after* 3 : 44 A.M.

CHAPTER IV

LOGARITHMS

107. In an equation $10^L = N$

the exponent L is called the *logarithm* of the number N, to the base 10.

For example,

$10^2 = 100$, hence 2 is the logarithm of 100, to the base 10.

We write, $2 = \log 100$.

$10^3 = 1000$, hence 3 is the logarithm of 1000, to the base 10.

We write, $3 = \log 1000$.

$10^{\frac{1}{2}} = 3.16227^+$ (verify this), hence .5 is the logarithm of 3.16227^+, to the base 10.

We write, $.5 = \log 3.16227^+$.

$10^{\frac{3}{2}} = 10^1(10)^{\frac{1}{2}} = 31.6227^+$, hence 1.5 is the logarithm of 31.6227^+, to the base 10.

We write, $1.5 = \log 31.6227^+$.

As in this chapter all logarithms are taken to the same base 10, no confusion will arise from the omission, hereafter, of the phrase "to the base 10."

Verify the following statements:

$10^3 = 1000$

$10^2 = 100$

$10^1 = 10$

$10^0 = 1$

$10^{-1} = .1$

$10^{-2} = .01$

$10^{-3} = .001$

$10^{\frac{5}{2}} = 100(10)^{\frac{1}{2}} = 316.227^+$

$10^{\frac{3}{2}} = 10(10)^{\frac{1}{2}} = 31.6227^+$

$10^{\frac{1}{2}} = 3.16227^+$

$10^{-\frac{1}{2}} = \dfrac{1}{10^{\frac{1}{2}}} = .316227^+$

$10^{-\frac{3}{2}} = \dfrac{1}{10^{\frac{3}{2}}} = .0316227^+$

$10^{-\frac{5}{2}} = \dfrac{1}{10^{\frac{5}{2}}} = .00316227^+$

Hence, by the definition of logarithms,

$3 = \log 1000$	$2.5 = \log 316.227^+$
$2 = \log 100$	$1.5 = \log 31.6227^+$
$1 = \log 10$	$.5 = \log 3.16227^+$
$0 = \log 1$	$-.5 = \log .316227^+$
$-1 = \log .1$	$-1.5 = \log .0316227^+$
$-2 = \log .01$	$-2.5 = \log .00316227^+$
$-3 = \log .001$	

LOGARITHMIC CURVE

108. On a piece of square paper as small as that in Fig. 15, it is not convenient to draw a curve that will exhibit the logarithms of numbers ranging from .001, the smallest, to 1000, the largest number considered above. The curve shows the logarithms of positive numbers between .1 and 50. The numbers are laid off on the x-axis, their respective logarithms on the y-axis.

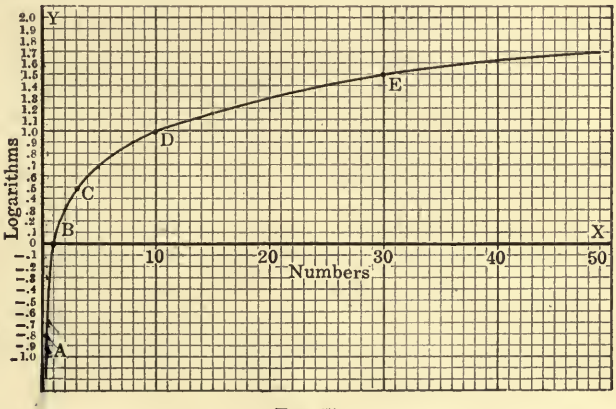

FIG. 15.

Observe that $-1 = \log .1$ locates the point A
$0 = \log 1$ locates the point B
$.5 = \log 3.16^+$ locates the point C
$1 = \log 10$ locates the point D
$1.5 = \log 31.6^+$ locates the point E

In drawing the curve, some additional values were used.

ORAL EXERCISES ON THE LOGARITHMIC CURVE

109. 1. Where does the curve cut the x-axis? How much is log 1?

2. What is the algebraic sign of the logarithms of all numbers larger than 1?

3. For what range of numbers are the logarithms negative?

4. Does the logarithmic curve extend to the left of the y-axis?

5. Do negative numbers have real logarithms?

6. Does the logarithm increase as a variable number increases?

By inspection of Fig. 15, find approximately the logarithms of the following numbers:

7. 5.	10. 35.	13. 2.
8. 15.	11. 45.	14. 4.
9. 25.	12. 50.	15. 6.

By inspection of Fig. 15, find approximately the numbers corresponding to the following logarithms:

16. 1.6.	20. 1.2.	24. .8.
17. 1.5.	21. 1.1.	25. .3.
18. 1.4.	22. 1.	26. 0.
19. 1.3.	23. .7.	27. $-.5$.

Find by inspection, how many times greater

28. log 4 is than log 2.	33. log 27 is than log 3.
29. log 8 is than log 2.	34. log 16 is than log 4.
30. log 16 is than log 2.	35. log 25 is than log 5.
31. log 32 is than log 2.	36. log 36 is than log 6.
32. log 9 is than log 3.	

Show by Fig. 15 that

37. $\log 50 = \log 5 + \log 10$.

38. $\log 40 = \log 5 + \log 8$.

39. $\log 42 = \log 6 + \log 7$.

40. $\log 48 - \log 8 = \log 6$.

41. $\log 24 - \log 4 = \log 6$.

42. $\log 32 - \log 4 = \log 8$.

ASSUMPTIONS

110. In drawing the logarithmic curve in Fig. 15 we located a few points and then drew a smooth curve through those points. This process is based on the following assumptions which admit of proof: (1) All positive numbers, whether rational or irrational, have logarithms; (2) In all cases, the logarithm of a number increases when the number itself increases.

FUNDAMENTAL THEOREM

111. In studying the logarithmic curve we noticed that $\log 50 = \log 5 + \log 10$.

This illustrates an important theorem.

Let N and N_1 be any two positive numbers.

Also, let $N = 10^L$,

and $N_1 = 10^{L_1}$.

Then their product is, $N \cdot N_1 = 10^{L+L_1}$.

By definition of logarithms, $L =$ logarithm of N,

and $L_1 =$ logarithm of N_1,

$L + L_1 =$ logarithm of $N \cdot N_1$.

Hence, the theorem,

The logarithm of the product of two positive numbers is equal to the sum of the logarithms of the numbers.

112. The integral part of a logarithm is called its *characteristic*, and the decimal part is called its *mantissa*.

For example, $\log 31.6227+ = 1.5$.
The characteristic of $\log 31.6227+$ is 1.
The mantissa of $\log 31.6227+$ is .5.

The tables of logarithms give only the mantissa; the characteristic can be supplied by two easy rules.

It has been found convenient to take the mantissas of all logarithms *positive*. The characteristic is positive in the logarithms of numbers larger than 1, and negative in the logarithms of numbers smaller than 1. That is, the characteristic of log 100 is positive; the characteristic of log .01 is negative. If the characteristic is negative, the negative sign is placed *over* the figure, as a reminder that the $-$ does *not* apply to the mantissa. Thus the characteristic of log .06 is $\bar{2}$.

For the purpose of deriving the rule for determining the characteristic of the logarithm of a number, we restate some of the relations in § 107:

$$1000 = 10^3 \qquad 1 = 10^0$$
$$100 = 10^2 \qquad .1 = 10^{-1}$$
$$10 = 10^1 \qquad .01 = 10^{-2}$$
$$1 = 10^0 \qquad .001 = 10^{-3}$$

Since 569.5 lies between 100 and 1000, its logarithm lies between 2 and 3.
That is, $\log 569.5 = 2 +$ a mantissa.
Since 85.6 lies between 10 and 100, $\log 85.6 = 1 +$ a mantissa.
Since 7.03 lies between 1 and 10, $\log 7.03 = 0 +$ a mantissa.
Since .673 lies between .1 and 1, $\log .673 = -1 +$ a mantissa.
Since .045 lies between .01 and .1, $\log .045 = -2 +$ a mantissa.
Since .0078 lies between .001 and .01, $\log .0078 = -3 +$ a mantissa.

By inspection of these relations we obtain the rule:

1. *If the* **first significant** *figure of a number is n places to the* $\left\{\begin{matrix} \textit{left} \\ \textit{right} \end{matrix}\right\}$ *of units' place, the characteristic of the logarithm is* $\left\{\begin{matrix} +n \\ -n \end{matrix}\right\}$.

The following illustrations will make this rule plainer:

									Characteristic
			Units' place						
$\dot{5}$	$\dot{6}$	9	.	5					2
	$\dot{8}$	5	.	6					1
		7	.	0	3				0
		0	.	$\dot{6}$	7	3			-1
		0	.	$\dot{0}$	$\dot{4}$	5			-2
		0	.	$\dot{0}$	$\dot{0}$	$\dot{7}$	8		-3

It will be seen that when a number is 10 or larger than 10, the first significant figure is to the *left* of units' place; if the number is less than 1, the first significant figure is to the *right* of units' place. This is shown by the dots placed above the digits; the number of dots placed over a number indicates the number of units in the characteristic.

EXERCISES

113. By inspection, tell the characteristics of the logarithms of the following numbers:

1. 123.	4. 17.754.	7. .091.	10. 2000.
2. 97.	5. .7894.	8. .00034.	11. .00005.
3. 2.345.	6. .005.	9. .0034.	12. 13.764.

114. The following is an important property possessed by logarithms constructed to the base 10:

If two numbers contain the same figures in the same order, but differ in the position of the decimal point, their logarithms to the base 10 *have the same mantissa.*

Suppose the two numbers are 7896.1 and 7.8961.

We have $7896.1 = 7.8961 \times 1000.$

By § 111, $\log 7896.1 = \log 7.8961 + \log 1000.$

But $\log 1000 = 3.$

Hence, $\log 7896.1 = \log 7.8961 + 3.$

Since the two logarithms differ by the integer 3, the decimal parts (the mantissas) of the logarithms must be the same.

For example, $\log 7896.1 = 3.8974.$

$\log 7.8961 = 0.8974.$

How the mantissa is found from the tables will be explained next.

FINDING LOGARITHMS

115. 1. Find the logarithm of 789.

The characteristic is 2.

The mantissa is taken from the table, p. 110. The column on the extreme left contains the first two significant figures of the number whose logarithm is sought. In this case we look for the figures 78. In the same row with 78, and in the column headed "9" (9 being the third digit in 789), is the number 8971; this is the mantissa sought, expressed to four decimal places.

Hence we have $\log 789 = 2.8971.$

2. Find the logarithm of .02738.

The characteristic is $\bar{2}$. The significant figures are 2738. On p. 109 look in the column on the extreme left for 27; in the column headed "3" and in the same row with 27 is the mantissa .4362.

Hence $\log .0273 = \bar{2}.4362.$ But we want $\log .02738.$ This is not found in this table and must be determined by a process called "interpolation."

The table is so constructed that interpolation is made easy. In the column on the extreme right, headed "789," we find, under the 8, and in the same row with 4362, the number 13.

Add 13 to 4362. We obtain 4375. This is the mantissa, obtained by interpolation.

Hence, $\log .02738 = \bar{2}.4375.$

116. It may be added that logarithmic computation in general is only approximate. With a four-place table like the one in this book, logarithms of numbers are found to four places only. Absolute accuracy is, therefore, out of the question. This fact does not materially diminish the value of logarithms, for the reason that in practical problems, results correct to three, four, or five decimal places are satisfactory. For instance, in the computation of the interest on a given sum, a result correct to the nearest cent is all we want.

117. **Logarithms**

No.	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4	8	12	17	21	25	29	33	37
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4	8	11	15	19	23	26	30	34
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3	7	10	14	17	21	24	28	31
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3	6	10	13	16	19	23	26	29
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3	6	9	12	15	18	21	24	27
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3	6	8	11	14	17	20	22	25
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3	5	8	11	13	16	18	21	24
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2	5	7	10	12	15	17	20	22
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2	5	7	9	12	14	16	19	21
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2	4	7	9	11	13	16	18	20
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7

Logarithms

No.	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4

EXERCISES

118. Find the logarithms of:

1. 870.	4. .01235.	7. 13.8964.
2. 63.	5. .003986.	8. 1089.
3. 708.	6. .7943.	9. 1008.

FINDING ANTILOGARITHMS

119. 1. Given $\log n = 1.2355$, find the antilogarithm n.

The number n, frequently called the *antilogarithm*, may be found by steps which are the reverse of those taken in finding the logarithm.

At first consider only the mantissa 2355. On page 109 find the mantissa 2355. It occurs in the same row as the number 17 on the extreme left, and in the column headed "2." The order of the figures in the answer is 172.

Where should the decimal point be? That is determined with the aid of the characteristic. All we need to do is to apply the rule relating to characteristics. Place the decimal point so that the characteristic of the resulting number is 1.

That number is 17.2. Thus, $n = 17.2$.

2. Given $\log n = \bar{2}.8764$, find the antilogarithm n.

Look in the table for the mantissa 8764. It cannot be found.

Look then for the next smaller mantissa. It is 8762; the order of figures in the antilogarithm of $\bar{2}.8762$ is 752. But we want the antilogarithm of $\bar{2}.8764$. This is found approximately by the following "interpolation."

Find the difference between 8762 and the given mantissa. This difference is 2.

Look in the same row with 8762, and in one of the right-hand columns for the number 2. You see it in the column headed "3" and also in the column headed "4."

Take either 3 or 4 as the figure found by interpolation. Which of these is the more accurate, we cannot tell from these tables. Take 3.

Write 3 after 752, and we have the required order of figures; namely, 7523.

Place the decimal point so that, by the rule, the characteristic of the resulting number is $\bar{2}$. We obtain $n = .07523$, nearly.

EXERCISES

120. Find the antilogarithms of the following:

1. 1.7931.	4. $\bar{1}.7605$.	7. $\bar{3}.4567$.	10. 3.6709.
2. 2.3181.	5. 0.9064.	8. 0.7064.	11. $\bar{3}.5374$.
3. 3.9876.	6. $\bar{2}.1907$.	9. 2.3706.	12. 5.9860.

121. 1. Using logarithms, find the product $n = 7893 \times 8796$.

By § 111, the logarithm of a product is the sum of the logarithms c the factors. We find

$$\begin{aligned} \log 7893 &= 3.8973 \\ \log 8796 &= \underline{3.9443} \\ \log n &= 7.8416 \end{aligned}$$

We find the antilogarithm $n = 69430000$.

Observe that this answer is only approximate. The true answer is 69426828. Closer approximations may be obtained by logarithms, by using a table containing figures to 5, 6, 7 or more places.

2. Find the product $x = .9062 \times .007362$.

The characteristics of the factors are $\bar{1}$ and $\bar{3}$. When logarithms are added or subtracted, it is easier to write a negative characteristic as the difference of two positive integers, thus: $\bar{1} = 9 - 10$, $\bar{3} = 7 - 10$.

Accordingly, we get $\log .9062 = 9.9572 - 10$

$\log .007362 = \underline{7.8670 - 10}$

Add $\log x = 17.8242 - 20$

Or $\log x = \bar{3}.8242$

$x = .006671$, nearly.

EXERCISES

122. Using logarithms, compute the following products:

1. 123×978.	4. 37.61×3.945.	7. 9999×7777.
2. 12.34×75.34.	5. 1479×6984.	8. 19283×87045.
3. $.0009638 \times 7894$.	6. 2673×7654.	9 93069×80493.

FUNDAMENTAL THEOREMS

123. In § 111, we proved the important theorem that

The logarithm of a product is equal to the sum of the logarithms of the factors.

We proceed to establish two other theorems that are no less fundamental.

The logarithm of a quotient is equal to the logarithm of the dividend minus the logarithm of the divisor.

The proof is similar to that of the first theorem.

Let N and N_1 be any two positive numbers.

Let also $$N = 10^L$$

and $$N_1 = 10^{L_1}.$$

Divide N by N_1, $$\frac{N}{N_1} = 10^{L-L_1}.$$

By definition of logarithms,

$$L = \text{logarithm of } N,$$

$$L_1 = \text{logarithm of } N_1,$$

and $$L - L_1 = \text{logarithm of } \frac{N}{N_1}.$$

Hence the theorem is proved.

The logarithm of a positive number with the exponent n is n times the logarithm of the number.

In this theorem n may be a positive integer or a positive fraction.

Let N be the number, and let

$$N = 10^L.$$

Raise both sides to the nth power,

$$N^n = (10^L)^n.$$

Simplify $$N^n = 10^{nL},$$

By the definition of logarithms,

$$L = \text{the logarithm of } N$$

and

$$nL = \text{the logarithm of } N^n.$$

Hence the theorem is proved.

If $n = \frac{1}{m}$, then this theorem gives the logarithm of $\sqrt[m]{N}$.

124. 1. Using logarithms, find the quotient $x = \frac{9876}{6987}$.

We find
$$\log 9876 = 3.9946$$
$$\log 6987 = \underline{3.8443}$$
$$\log x = 0.1503$$
The antilogarithm
$$x = 1.414, \text{ nearly.}$$

2. Find $x = (8.786)^5$.

We find
$$\log 8.786 = 0.9438$$
Multiply by 5,
$$\underline{\qquad 5}$$
$$\log x = 4.7190$$
$$x = 52360, \text{ approximately.}$$

3. Compute $x = (.01237)^{\frac{7}{3}}$.

The characteristic of .01237 is $\bar{2}$; we write it $8 - 10$.

Hence,
$$\log .01237 = 8.0923 - 10$$
Multiply by 7,
$$\underline{\qquad\qquad 7}$$
$$56.6461 - 70$$

Divide by 3. Since $70 \div 3$ introduces a fraction, it is more convenient to write the characteristic, $56 - 70$, in the form $16 - 30$.

The logarithm is then
$$16.6461 - 30.$$
Dividing by 3,
$$\log x = 5.5487 - 10.$$
$$x = .00003537, \text{ nearly.}$$

EXERCISES

125. Evaluate the following by means of the four-place table of logarithms:

1. 6.823×2.315.
2. 46420×27.49.
3. $32.25 \div 5.923$.
4. $2293 \times 4489 \div 7895$.
5. $.943 \times 9855 \div .0896$.
6. $(2.519)^4$.

7. $(.007658)^3$.

8. $\sqrt[20]{4978000000}$.

9. $\sqrt[3]{752} \times \sqrt[5]{3450}$.

10. $(1.53)^{\frac{2}{3}}$.

11. $(3.197)^{\frac{4}{5}}$.

12. $\sqrt[7]{.03276}$.

13. $\dfrac{98.6 \times 7.639}{.0964}$.

14. $\dfrac{(273)^2(743)^3}{(897)^4}$.

15. $\dfrac{\sqrt{3.598} \cdot \sqrt[3]{479}}{\sqrt{93.4}}$.

PROBLEMS

126. In the following problems let $\pi = 3.1416$, R = radius.

1. Find the length of a circle (*i.e.* the circumference) whose radius is 3.17 ft.

2. Find the area of a circle whose radius is 9.795 in. (Area $= \pi R^2$).

3. Find the area of a circle whose radius is 7.891 cm.

4. Compute the curved surface of a right circular cylinder whose height is 7.9 in. and whose base has a diameter of 79.86 in.

5. Calculate the volume of a sphere whose radius is 2.97 yd.

$$\text{Volume of a sphere} = \frac{4\,\pi R^3}{3}.$$

6. What is the area of the surface of a sphere when $R = 3.7$ in.?

$$\text{Area of spherical surface} = 4\,\pi R^2.$$

7. Compute the leg of a right triangle when the hypotenuse is 25.97 in. and the other leg is 17.8 in.

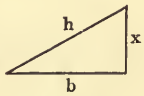

FIG. 16.

In Fig. 16, $x = \sqrt{h^2 - b^2} = \sqrt{(h + b)(h - b)}$.

8. Compute the area of a triangle whose sides are, respectively, 17.89, 20.96, 19.78.

Area of triangle $= \sqrt{s\,(s - a)\,(s - b)\,(s - c)}$, where a, b, c are the sides of the triangle and s is *half* the sum of the sides.

9. In a right triangle the hypotenuse is 9.675, one leg is 7.98; find the length of the other leg.

10. Find the area of the surface of a hemispherical dome, the diameter of which is 40 feet.

11. A triangular piece of ground measures along the edges 315 yd., 541 yd., 479 yd. Find the area.

12. The volume of a pyramid is one third the product of its base and altitude. What is the volume of a pyramid whose height is 7.82 ft. and the sides of whose triangular base are 6.92 ft., 5.83 ft., and 4.91 ft.?

EXPONENTIAL EQUATIONS

127. An *exponential equation* is one in which the unknown occurs in an exponent.

Thus, $12^x=15$, $a^{2x+3}=b$ are exponential equations which can be solved with the aid of logarithms.

Since $\log 12^x = x \log 12$,

we obtain from $12^x=15$, $x \log 12 = \log 15$.

Divide both sides by log 12, $x = \dfrac{\log 15}{\log 12}$.

Finding the logarithms, we obtain $x = \dfrac{1.1761}{1.0792}$.

$\dfrac{1.1761}{1.0792}$ is computed like any other quotient.

$$\log 1.1761 = 0.0705$$
$$\log 1.0792 = \underline{0.0332}$$

Subtracting, $\log x = 0.0373$

$$x = 1.090, \text{ nearly.}$$

Note that in computations that are only approximate the answer 1.090 receives a different interpretation than the answer 1.09. The difference is this: the answer 1.090 shows that an effort was made to ascertain the third decimal figure and that it was found to be 0. On the other hand, 1.09 means that no effort was made to ascertain the value of the third figure. Consequently 1.090 signifies an answer carried to 3 decimals, while 1.09 signifies an answer carried to but 2 decimals.

EXERCISES

128. Solve for x:

1. $7^x = 17$.
2. $125^x = 98$.
3. $9^{x+1} = 87$.
4. $15^{2x} = 25$.
5. $25^{2x+3} = 795$.
6. $9^{\frac{x}{5}} = 13$.

PROBLEMS

129. 1. In what time will \$ 1 double itself at 4 %, interest compounded annually?

At the end of the first year the interest is \$ .04, the amount is \$ 1.04.

At the end of the second year the interest is \$ (1.04)(.04), the amount is \$ 1.04 + (1.04)(.04) = \$ $(1.04)^2$.

At the end of the third year the interest is \$ $(1.04)^2$ (.04), the amount is \$ $(1.04)^3$.

At the end of x years the amount will be \$ $(1.04)^x$.

We have the equation, $(1.04)^x = 2$.

$$x = 17.7 \text{ nearly.}$$

2. In what time will \$ 1 treble itself at 4 %, interest compounded annually?

3. A sum of \$ 1000 bears 5 % interest compounded annually. What is the amount after 20 years?

4. Compute the compound interest on \$ 879 at 4 % for 13 years.

HISTORICAL NOTE

130. Logarithms constitute one of the most useful inventions in mathematics. A great French astronomer once said that logarithms "by shortening the labors, doubled the life of the astronomer." Logarithms were invented by John Napier, Baron of Merchiston, in Scotland, who in 1614 published a table of logarithms and described their use. An independent inventor of logarithms was the Swiss watchmaker and astronomer, Joost Bürgi, who published a table in 1620, at a time when Napier's logarithms were already known and admired throughout Europe. Napier's and Bürgi's logarithms differed somewhat from each other and from the logarithms described in this book. Our system of logarithms to the base

10 was designed by John Napier and his friend Henry Briggs, conjointly, before the year 1617. At one time Napier lived in a beautiful castle on the banks of the Endrick. Yet even in such surroundings he was not free from annoyances which hindered intellectual work. On the opposite side of the river was a lint mill, and its clack greatly disturbed Napier. He sometimes desired the miller to stop the mill so that the train of his ideas might not be interrupted. Much of Napier's time was taken up in the management of his estate. Joost Bürgi was not rich, like Napier; the necessity of earning a livelihood greatly hampered his scientific work. But both Bürgi and Napier were men who loved mathematics and were able to achieve great results in spite of many hindrances.

CHAPTER V

QUADRATIC EQUATIONS AND THEIR PROPERTIES

REVIEW

131. In solving a quadratic equation of the form $ax^2 + bx + c = 0$ by "completing the square," the first step in our explanation was to divide both sides of the equation by the coefficient of x^2. There are other methods of procedure which possess certain advantages; fractions having large denominators may be avoided, or, fractions may be avoided altogether. These methods will be explained in § 191.

In the following exercises solve by any method. To find the square roots of numbers, use the tables in § 197.

1. $5x^2 + 20x = 17.$
2. $ax^2 + abx + 4ab = 0.$
3. $5x^2 + 21x = 30.$
4. $ax^2 + abx + a^2 = b^2.$
5. $35x - 3x^2 = 11.$
6. $3ax^2 + 6x - 5a^2 = 0.$
7. $40x - 3x^2 = 12.$
8. $(a+b)x^2 + 2(a-b)x + a = 0.$

EQUATIONS QUADRATIC IN FORM

132. Equations like $x^{2n} + ax^n + b = 0$ are said to be quadratic in form, because the unknown x occurs only with the exponents n and $2n$, where one exponent is *twice* the other. Equations of this form can be solved by the methods used in solving ordinary quadratic equations.

1. Solve $x^4 - 7x^2 + 10 = 0.$

Factoring, $(x^2 - 5)(x^2 - 2) = 0.$

$x^2 - 5 = 0,\ x^2 - 2 = 0.$

$x = \pm\sqrt{5},\ x = \pm\sqrt{2}.$

From the tables in § 197 we obtain $x = \pm 2.236$, $x = \pm 1.414$, approximately.

2. Solve $$x^6 + 3x^3 + 1 = 0.$$

Complete the square, $x^6 + 3x^3 + \frac{9}{4} = -1 + \frac{9}{4} = \frac{5}{4}$.

$$x^3 + \tfrac{3}{2} = \pm\frac{\sqrt{5}}{2}.$$

$$x^3 = -\tfrac{3}{2} \pm \frac{\sqrt{5}}{2}.$$

$$x = \sqrt[3]{-\tfrac{3}{2} \pm \frac{\sqrt{5}}{2}}.$$

As there are in general three cube roots to a number, and there are in this case two different numbers under the radical sign, it is seen that there are six roots of the given equation. Of these, four are imaginary. As we wrote the answer, only the *two principal roots* are indicated.

EXERCISES

133. Solve:

1. $x^4 + 6x^2 - 16 = 0.$
2. $x^8 - 2x^4 + 1 = 0.$
3. $x^6 + 3x^3 + 1 = 0.$
4. $x^4 - 8x^2 - 20 = 0.$
5. $2x^4 - 7x^2 - 30 = 0.$
6. $6m^6 - 23m^3 + 20 = 0.$
7. $y^6 - 2y^3 - 143 = 0.$
8. $z^4 = 256.$
9. $y^{2n} - 5y^n + 6 = 0.$

RELATIONS BETWEEN ROOTS AND COEFFICIENTS

134. From the solution of $ax^2 + bx + c = 0$, we obtain

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

If we designate the two roots by x_1 and x_2, we have

$$x_1 = -\frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}$$

and $$x_2 = -\frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}.$$

The sum of the roots is

$$x_1 + x_2 = \frac{-2b}{2a} = -\frac{b}{a}.$$

The product of the roots is

$$x_1 x_2 = \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right)$$

$$= \frac{b^2 - (b^2 - 4ac)}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a}.$$

If we divide both sides of $ax^2 + bx + c = 0$ by a, we obtain

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0,$$

in which the coefficient of x^2 is unity. We see that, in the last equation,

(1) *The sum of the roots is the coefficient of x with the sign changed;* and

(2) *The product of the roots is the absolute term.*

To illustrate: Solve $x^2 + 5x + 6 = 0$.

The roots are -2 and -3. Their sum is -5, the coefficient of x with its sign changed; their product is $+6$, the absolute term.

From this relation between the coefficients and the roots, we may form an equation, if the roots are known.

Form the equation whose roots are -1 and 6.

We assume that, in the required equation, the coefficient of x^2 is 1.

Then, $x_1 + x_2 = 5$, the coefficient of x with its sign changed; $x_1 x_2 = -6$, the absolute term.

Hence, the equation is $x^2 - 5x - 6 = 0$.

Or, we might write the equation thus:

$$x^2 - (-1 + 6)x + (-1 \cdot 6) = 0.$$

Whence, $$x^2 - 5x - 6 = 0.$$

135. Form the equation whose roots are:

1. $+5, +2.$
2. $-3, -8.$
3. $4, -6.$
4. $7, 0.$
5. $2\frac{1}{2}, 3\frac{1}{3}.$
6. $-\frac{1}{5}, -\frac{1}{6}.$
7. $-a, -b.$
8. $m^2, -3m^2.$
9. $\sqrt{2}, -\sqrt{3}.$
10. $1+\sqrt{2}, 1-\sqrt{2}.$
11. $-\sqrt{5}, -\sqrt{3}.$
12. $\frac{1}{2}\sqrt{2}, -\frac{1}{2}\sqrt{2}.$

If you know one root of a given quadratic equation, how can you calculate the other: (1) from the coefficient of x, (2) from the absolute term, (3) by the Factor Theorem? Use each of these methods in turn, to find the second root in each of the following equations:

13. $x^2 - 8.7x + 17.6 = 0$, one root 5.5.
14. $x^2 - 2.1x + .9 = 0$, one root 1.5.
15. $x^2 - 1.7x - 4.8 = 0$, one root 3.2.
16. $x^2 + 11.8x + 34.17 = 0$, one root -5.1.
17. $x^2 + 3.6x - 25.92 = 0$, one root -7.2.
18. $x^2 - .085x + .00175 = 0$, one root .05.

NATURE OF THE ROOTS

136. We shall discuss the nature of the roots of the quadratic equation $ax^2 + bx + c = 0$, in which the letters a, b, c are assumed to be real and rational numbers. When some of these letters are imaginary or irrational, the conclusions which we shall draw do not necessarily hold. The solution of $ax^2 + bx + c = 0$ is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The nature of the roots depends upon $b^2 - 4ac$, which is called the *discriminant.*

I. When $b^2 - 4ac = 0$, the roots are *real* and *equal.* Explain.

II. When $b^2 - 4ac > 0$, the roots are *real* and *unequal.* If $b^2 - 4ac$ is a perfect square, both roots are *rational;* if

$b^2 - 4\,ac$ is not a perfect square, both roots are *irrational.* Why?

III. When $b^2 - 4\,ac < 0$, both roots are *imaginary.* Explain.

To illustrate: 1. In $x^2 - 10\,x + 25 = 0$, $b^2 - 4\,ac = 100 - 100 = 0$. Hence the roots are *real*, *rational*, and *equal.* The roots are 5, 5.

2. In $x^2 - 3\,x - 28 = 0$, $b^2 - 4\,ac = 9 + 112 = 121$. Hence, the roots are *real* and *unequal;* since 121 is a perfect square, both roots are *rational.*

3. In $x^2 - 2\,x + 6 = 0$, $b^2 - 4\,ac = 4 - 24 = -20$. Hence both roots are *imaginary.*

EXERCISES

137. Determine, without solving, the nature of the roots of:

1. $x^2 - x - 12 = 0.$
2. $x^2 + 8\,x - 16 = 0.$
3. $x^2 - 8\,x - 25 = 0.$
4. $5\,x^2 + 7\,x + 1 = 0.$
5. $2\,x^2 + 3\,x = 8.$
6. $7\,x^2 - 12\,x + 4 = 0.$
7. $4\,x^2 - 12\,x = -9.$
8. $4\,x^2 - 12\,x - 55 = 0.$
9. $x^2 - 2\,x + 3 = 0.$
10. $3\,x^2 + 13\,x + 1 = 0.$

For what values of a are the roots of the following equations equal? Real and unequal? Imaginary?

11. $ax^2 + 4\,x = -1.$
12. $x^2 + ax + 5 = 0.$
13. $3\,x^2 + 3\,x + a = 0.$
14. $ax^2 + ax + 2 = 0.$

GRAPH OF THE QUADRATIC EQUATION, $ax^2 + bx + c = y$

138. The value of the expression, $ax^2 + bx + c$, for given values of a, b, and c, depends upon the value of the *variable* x. For that reason the expression is called a *function of* x. If we write $ax^2 + bx + c = y$, then for every value of x there is a corresponding value of y.

The values of x and the corresponding values of y may be taken as the coördinates of points on the graph of the equation $ax^2 + bx + c = y$.

1. Draw the graph of $x^2 - 4x - 5 = y$.

$x^2 - 4x - 5 = y$

x	y	Point
7	16	A
6	7	B
5	0	C
4	-5	D
3	-8	E
2	-9	F
1	-8	G
0	-5	H
-1	0	I
-2	7	J
-3	16	K

Thus it will be seen (Fig. 17) that the graph crosses the x-axis at the points (5, 0) and $(-1, 0)$. Upon solving $x^2 - 4x - 5 = 0$, we find that $x = +5$ and -1, and these are the values of x which make $y = 0$.

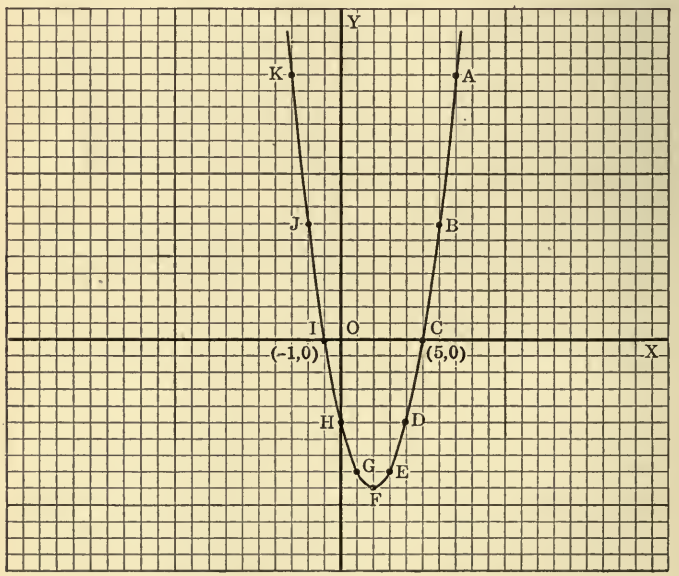

FIG. 17.

2. Draw the graph of $x^2 - 6x + 9 = y$.

$x^2 - 6x + 9 = y$

x	y	Point
6	9	*A*
5	4	*B*
4	1	*C*
3	0	*D*
2	1	*E*
1	4	*F*
0	9	*G*

In Fig. 18 it is seen that the graph touches the x-axis, but does not cross it. The graph turns and proceeds upward.

Instead of two points of *intersection* with the x-axis, as in Fig. 17, there is in Fig. 18 one point of *contact;* the two points have united in a single point. Evidently there are two roots at that point. Upon solving the equation we find that $x = 3, 3$; 3 is called a *double root.*

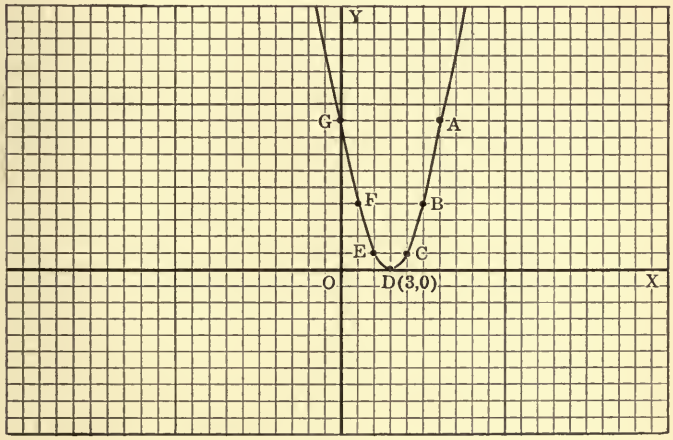

FIG. 18.

3. Draw the graph of $x^2 - 2x + 3 = y$.

$x^2 - 2x + 3 = y$

x	y	Point
5	18	*A*
4	11	*B*
3	6	*C*
2	3	*D*
1	2	*E*
0	3	*F*
-1	6	*G*
-2	11	*H*
-3	18	*I*

Fig. 19 illustrates the fact that not every graph cuts the x-axis.

If we solve $x^2 - 2x + 3 = 0$, we find that $x = 1 \pm \sqrt{-2}$. Now $\sqrt{-2}$ is an *imaginary* number; therefore, there is no real value of x which will make $x^2 - 2x + 3$ equal to 0.

The graph makes this fact evident at a glance.

We see that the two roots of the equation $ax^2 + bx + c = 0$ may be real and unequal, as in Fig. 17, real and equal, as in Fig. 18, or imaginary, as in Fig. 19.

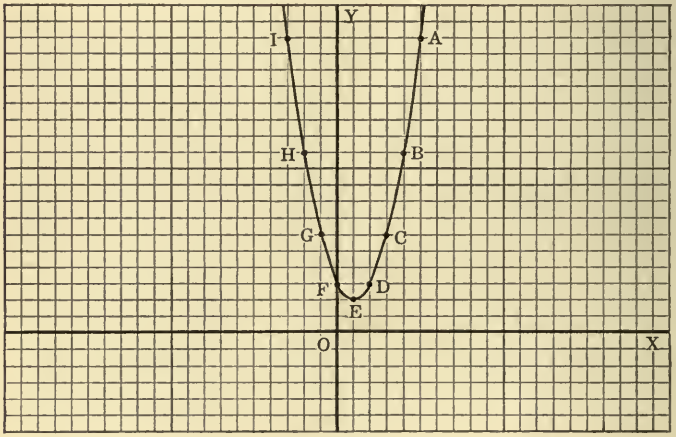

FIG. 19.

CHAPTER VI

SYSTEMS OF EQUATIONS SOLVABLE BY QUADRATICS

I. ONE EQUATION IN THE SYSTEM IS LINEAR

139. Solve

$$x^2 + y^2 = 25. \qquad (1)$$

$$x - y = 1, \qquad (2)$$

Let us solve this system, first graphically, then algebraically.

The equation $x - y = 1$ is linear and is represented by a straight line. The graph of $x^2 + y^2 = 25$ is a circle.

$x - y = 1$

x	y	Point
0	-1	A
1	0	B

$x^2 + y^2 = 25$

x	y	Point
5	0	C
4	± 3	D, E
3	± 4	F, G
0	± 5	H, I
-3	± 4	K, L
-4	± 3	M, N
-5	0	O

Fig. 20 shows the graph of $x^2 + y^2 = 25$, (1), and $x - y = 1$, (2). The line (2) intersects the circle in two points $(-3, -4)$ and $(4, 3)$.

The coördinates of these points satisfy both equations and are therefore the values sought; namely,

$$x = 4 \text{ and } y = 3; \quad x = -3 \text{ and } y = -4.$$

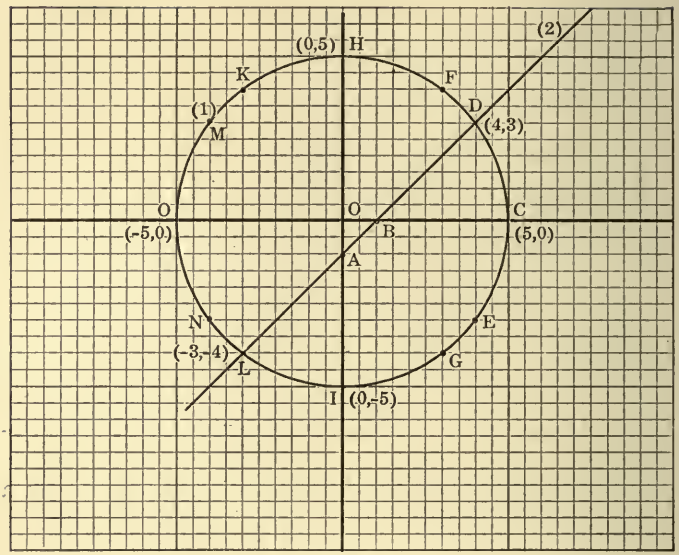

FIG. 20.

140. Observe that x and y, when subject to the *one* condition $x^2 + y^2 = 25$, are *variables*, because they are capable of taking on successively a never ending series of different values. For the same reason x and y are also *variables* when subject to the *one* condition $x - y = 1$. But x and y can no longer assume successively a never ending series of different values, when they are subject at the same time to *both* conditions, $\left\{\begin{array}{l} x^2 + y^2 = 25 \\ x - y = 1 \end{array}\right\}$. Hence x and y are now no longer *variables*, but *constants*. It is the purpose of the solution to find the values of these constants.

In drawing graphs we are dealing with *variables;* in finding the points of intersection of these graphs, we are finding the values of *constants* which, at the outset, are unknown.

141. The algebraic solution of

$$x^2 + y^2 = 25 \qquad (1)$$
$$x - y = 1 \qquad (2)$$

can be effected easiest by the method of *substitution.*

From (2), $x = 1 + y.$

Substitute in (1), $1 + 2y + y^2 + y^2 = 25.$

Whence, $y^2 + y - 12 = 0.$

$(y + 4)(y - 3) = 0.$

$y = -4$, or $+3$.

Substitute in (2), $x = -3$, or $+4$.

The two sets of roots are, $x = -3$, $y = -4$, and $x = 4$, $y = +3$.

In checking, substitute the values of x and y in *both* of the given equations; the answers might be wrong, yet might satisfy *one* of the two equations. For instance, $x = 3$, $y = 2$ will satisfy (2), but not (1).

If in Fig. 20 the line should move, parallel to its present position, until it were tangent to the circle, we would have just one set of values satisfying the pair of equations. Show that this is the case, if the equations are $x^2 + y^2 = 25$ and $x - y = 5\sqrt{2}$.

If the line should not intersect the circle, nor be tangent to it, we would have a set of imaginary values satisfying both equations. Show that this is the case, if the equations are $x^2 + y^2 = 25$, $x - y = 10$.

142. Instead of elimination by substitution, some special device is frequently used.

1. Solve

$$x^2 - y^2 = 5, \qquad (1)$$
$$x - y = -5. \qquad (2)$$

Divide (1) by (2),

$$x + y = -1 \qquad (3)$$
$$x - y = -5 \qquad (2)$$

Add (2) and (3),

$$2x = -6$$
$$x = -3$$

Subtract (2) from (3),

$$2y = 4$$
$$y = 2. \quad \textit{Ans.}\ x = -3,\ y = 2.$$

$x - y = -5$

x	y	Point
0	5	A
-5	0	B

$x^2 - y^2 = 5$

x	y	Points
$\pm 2.2^+$	0	A, A_1
3	± 2	B, B_1
-3	± 2	C, C_1
4	$\pm 3.3^+$	D, D_1
-4	$\pm 3.3^+$	E, E_1

Explain how Fig. 21 exhibits the values of x and y which satisfy equations (1) and (2).

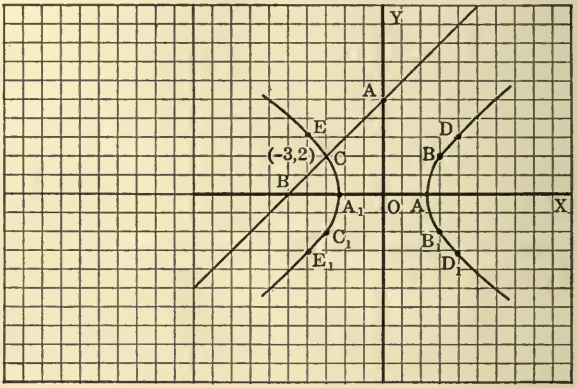

FIG. 21.

2. Solve $\quad x - y = 6, \qquad (1)$

$$xy = -5. \qquad (2)$$

Square (1), $\quad x^2 - 2xy + y^2 = 36. \qquad (3)$

(2) · 4, $\quad 4xy = -20. \qquad (4)$

Add (3) and (4), $\quad x^2 + 2xy + y^2 = 16. \qquad (5)$

Extract square root, $\quad x + y = \pm 4, \qquad (6)$

$$x - y = 6. \qquad (1)$$

Add (6) and (1), $\quad 2x = 10, 2,$

$$x = 5, 1.$$

Subtract (1) from (6), $\quad 2y = -2, -10,$

$$y = -1, -5.$$

The sets of roots are $\quad x = 5, y = -1\,;\ x = 1, y = -5.$

3. Solve
$$x^2 + y^2 = 20, \quad (1)$$
$$xy = 8. \quad (2)$$

Multiply (2) by 2, $\quad 2\,xy = 16. \quad (3)$

Add (1) and (3), $\quad x^2 + 2\,xy + y^2 = 36. \quad (4)$

Subtract (1) and (3), $\quad x^2 - 2\,xy + y^2 = 4. \quad (5)$

Extract square root of (4) and (5),
$$x + y = \pm 6, \quad (6)$$
$$x - y = \pm 2. \quad (7)$$

Whence,
$$2\,x = \pm 8,\ \pm 4,$$
$$x = \pm 4,\ \pm 2.$$
$$2\,y = \pm 4,\ \pm 8,$$
$$y = \pm 2,\ \pm 4.$$

The sets of roots are, $x = \pm 4,\ y = \pm 2;\ x = \pm 2,\ y = \pm 4.$

The graphic solution is exhibited in Fig. 22.

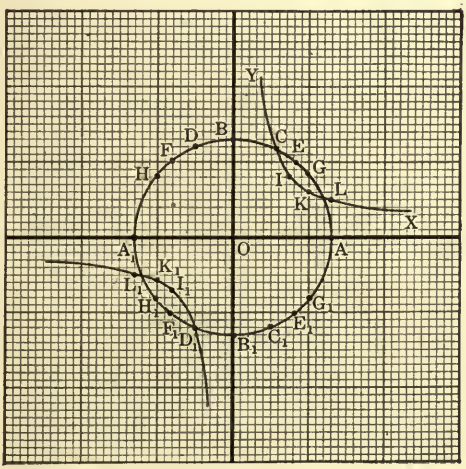

FIG. 22.

$x^2 + y^2 = 20$

x	y	Point
± 4.47	0	A, A_1
0	± 4.47	B, B_1
2	± 4	C, C_1
-2	± 4	D, D_1
3	$\pm 3.3^+$	E, E_1
-3	$\pm 3.3^+$	F, F_1
4	± 2	G, G_1
-4	± 2	H, H_1

$xy = 8$

x	y	Point
± 2	± 4	C, D_1
± 3	$\pm 2\frac{2}{3}$	I, I_1
± 4	± 2	K, K_1
± 5	$\pm 1\frac{3}{5}$	L, L_1

4. Solve
$$x^3 - y^3 = 91, \quad (1)$$
$$x - y = 7. \quad (2)$$

Divide (1) by (2), $x^2 + xy + y^2 = 13.$ (3)

Square (2), $x^2 - 2\,xy + y^2 = 49.$ (4)

Subtract (4) from (3), $3\,xy = -36,$

$xy = -12.$ (5)

Add (5) and (3), $x^2 + 2\,xy + y^2 = 1.$ (6)

Extract square root $x + y = \pm 1.$ (7)

$x - y = 7.$ (2)

Whence $2\,x = 8, 6,$

$x = 4, 3.$

And $2\,y = -6, -8,$

$y = -3, -4.$

The sets of roots are $x = 4, y = -3\,;\ x = 3, y = -4.$

5. Solve
$$x^4 + y^4 = 97, \quad (1)$$
$$x + y = 5. \quad (2)$$

Let $x = u + v\,;\ y = u - v.$

Substitute in (1) and (2),

$u^4 + 4\,u^3v + 6\,u^2v^2 + 4\,uv^3 + v^4 + u^4 - 4\,u^3v + 6\,u^2v^2 - 4\,uv^3 + v^4 = 97.$ (3)

Whence, $2\,u^4 + 12\,u^2v^2 + 2\,v^4 = 97,$ (4)

and $u + v + u - v = 5.$ (5)

Whence, $2\,u = 5,$

$u = \frac{5}{2}.$

Substitute $u = \frac{5}{2}$ in (4), $\qquad \frac{625}{8} + 75\,v^2 + 2\,v^4 = 97. \qquad (6)$

Whence, $\qquad 16\,v^4 + 600\,v^2 = 151.$

Completing the square, $\qquad 16\,v^4 + 600\,v^2 + 5625 = 5776,$

$$4\,v^2 + 75 = \pm 76,$$

$$4\,v^2 = 1,\ -151,$$

$$v^2 = \tfrac{1}{4},\ -\tfrac{151}{4},$$

$$v = \pm\tfrac{1}{2},\ \pm\tfrac{1}{2}\sqrt{-151},$$

$$u = \tfrac{5}{2}.$$

Hence, $\qquad x = 3,\ 2,\ \tfrac{5}{2} \pm \tfrac{1}{2}\sqrt{-151},$

$$y = 2,\ 3,\ \tfrac{5}{2} \mp \tfrac{1}{2}\sqrt{-151}.$$

The sets of roots are

$$x = 3,\ y = 2\,;\ x = 2,\ y = 3;\ x = \tfrac{5}{2} \pm \tfrac{1}{2}\sqrt{-151},\ y = \tfrac{5}{2} \mp \tfrac{1}{2}\sqrt{-151}.$$

Check each set of real roots.

The check for the imaginary roots is as follows:

(1) $$\tfrac{625}{16} \pm \tfrac{125}{4}\sqrt{-151} - \tfrac{11325}{8} \mp \tfrac{755}{4}\sqrt{-151} + \tfrac{22801}{16} + \tfrac{625}{16}$$

$$\mp \tfrac{125}{4}\sqrt{-151} - \tfrac{11325}{8} \pm \tfrac{755}{4}\sqrt{-151} + \tfrac{22801}{16} = 97,$$

$$\tfrac{1552}{16} = 97,$$

$$97 = 97.$$

(2) $$\tfrac{5}{2} \pm \tfrac{1}{2}\sqrt{-151} + \tfrac{5}{2} \mp \tfrac{1}{2}\sqrt{-151} = 5,$$

$$5 = 5.$$

143. The equations $x^4 + y^4 = 97$ and $x + y = 5$ are called symmetrical equations, because they remain unaltered when x is written for y and y is written for x.

An expression is *symmetrical* with respect to two or more letters, if it remains unaltered when the letters are interchanged.

Every system of two equations that are *symmetrical*, or symmetrical *except for the signs*, can be solved by assuming $x = u + v$ and $y = u - v$. Instances of equations symmetrical except for the signs are $x^4 - y^4 = a$, $x - y = b$.

II. A SYSTEM OF TWO EQUATIONS IN X AND Y, BOTH QUADRATIC

144. *Any two equations of the form $ax^2+by^2=c$, or $ax^2+bxy=c$, or $ax^2+bxy+cy^2=d$, can be solved by assuming $y=vx$, and determining the constant v.*

It is seen that in these equations all the terms which contain x and y are of the second degree in x and y. Some authors call these equations *homogeneous*. We avoid this term for the reason that the word *homogeneous* is more commonly used to designate equations like $ax^2+bxy+cy^2=0$, in which *every* term of the equation contains x and y, and is of the same degree in x and y. See § 8.

Solve $$x^2+xy=-1, \tag{1}$$
$$y^2-2xy=8. \tag{2}$$

Let $y=vx$.

Then $$x^2+vx^2=-1, \tag{3}$$
and $$v^2x^2-2vx^2=8. \tag{4}$$

From (3), $(1+v)x^2=-1$, and $x^2=\dfrac{-1}{1+v}$. (5)

From (4), $(v^2-2v)x^2=8$, and $x^2=\dfrac{8}{v^2-2v}$. (6)

$$\therefore \frac{-1}{1+v}=\frac{8}{v^2-2v}. \tag{7}$$

Whence, $$-v^2+2v=8+8v, \tag{8}$$
and $$v^2+6v+8=0, \tag{9}$$
and $$(v+4)(v+2)=0.$$

Then $$v=-4,\ -2.$$

Substitute in (5), $$x^2=\frac{-1}{-3}=\frac{1}{3},\ \text{and } x^2=\frac{-1}{-1}=1,$$
$$x=\pm\tfrac{1}{3}\sqrt{3},\ \pm 1,$$
$$y=vx=\mp\tfrac{4}{3}\sqrt{3},\ \mp 2.$$

The two sets of roots are $x=\pm\frac{1}{3}\sqrt{3}$, $y=\mp\frac{4}{3}\sqrt{3}$; $x=\pm 1$, $y=\mp 2$.

To find the value of y care must be taken to use the value of v with that value of x which was obtained by substituting the value of v.

When -4 was substituted for v in (5), yielding $x=\pm\frac{1}{3}\sqrt{3}$, the -4 must be multiplied by $\pm\frac{1}{3}\sqrt{3}$ (since $y=vx$), to obtain the corresponding value of y.

Hence the values of x and y must be carefully paired as shown above.

When $\pm$ and $\mp$ are used in the answers, the upper signs go together and the lower signs go together.

145. The method just explained possesses the great advantage of always yielding results. Very often, however, much shorter solutions can be given by special devices. Frequently a third equation can be derived from the two given equations which is simpler than one or both of those given. The solution is then obtained by the use of this simpler equation along with one of the original ones.

For example, solve $x^2 - 3xy = -143,$ (1)

$y^2 + xy = 168.$ (2)

By adding (1) and (2) the simpler equation is obtained,

$$x^2 - 2xy + y^2 = 25.$$

Extract the square root of both sides, $x - y = \pm 5.$ (3)

(3) is a *linear* equation. It gives $x = \pm 5 + y.$

Substitute in (2), $y^2 \pm 5y + y^2 = 168,$

$$2y^2 \pm 5y = 168.$$

Taking the upper sign in $\pm 5y$, $y = \dfrac{-5 \pm \sqrt{1369}}{4},$

$$y = \frac{-5 \pm 37}{4}.$$

Taking the lower sign in $\pm 5y$, $y = \dfrac{+5 \pm 37}{4}.$

Hence, $y = 8, -8, \frac{21}{2}, -\frac{21}{2}.$

Substitute the values of y in (2), $x = 13, -13, \frac{11}{2}, -\frac{11}{2}.$

The sets of roots are $x = \pm 13,\ y = \pm 8;\ x = \pm \frac{11}{2},\ y = \pm \frac{21}{2}.$

POSSIBILITY OF SOLUTION BY QUADRATICS

146. It is readily seen that when, in a system of two equations, one equation is of the second degree and the other is of the first degree or linear, a solution may always be obtained by quadratics.

If, however, both equations are quadratics, this is usually not the case. Only special types of quadratic equations, such as have been studied in this chapter, and others of similar

character, admit of being solved by quadratics. The general case is far more complicated. Given

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

and $$a_1x^2 + b_1xy + c_1y^2 + d_1x + e_1y + f_1 = 0,$$

to find x and y. If y is eliminated by substitution, the result is a quartic equation, which is of the fourth degree and cannot be solved by quadratics, except in special cases. It is shown in more advanced algebras that the algebraic solution of a quartic equation depends in general upon the solution of a certain cubic equation or equation of the third degree. The solution of cubic and quartic equations is not explained in this book.

The given equations of the second degree in x and y can be solved by quadratics whenever the auxiliary cubic equation here mentioned possesses a *rational* root. This rational root can be found by the factor theorem (see § 78); the other two roots of the cubic can then be found by quadratics, as can also the four sets of roots of the given equations. (See F. Cajori, *Theory of Equations*, New York, 1914, pp. 71–73.)

EXERCISES

147. Solve

1. $2x^2 - 3xy = 5,$
   $x - y = 2.$

2. $x + y = 1,$
   $x^2 + y^2 = 85.$

3. $\frac{1}{x} - \frac{1}{y} = -\frac{8}{15},$
   $\frac{1}{x^2} + \frac{1}{y^2} = \frac{34}{225}.$

4. $x^3 + y^3 = 28,$
   $x + y = -2.$

5. $xy = 45,$
   $x + y = 14.$

6. $x^2 + 3xy = -5,$
   $2xy + y^2 = -16.$

7. $x + y = a,$
   $4xy - a^2 = -b^2.$

8. $x - y = 2n,$
   $x^2 - y^2 = 4mn.$

9. $x^4 - y^4 = 2401,$
   $x^2 + y^2 = 49.$

10. $x^2 + 7xy = -104,$
    $5xy - y^2 = -129.$

11. $x^5 + y^5 = 33,$
    $x + y = 3.$

12. $x - y = 1,$
$\frac{x}{y} + \frac{y}{x} = 2\frac{1}{20}.$

13. $2x - 3y = 0,$
$7xy + 3x - 4y = 43.$

14. $x^3 - y^3 = m^3,$
$x - y = m.$

15. $x^3 + y^3 = 109,$
$x^2y + xy^2 = -36.$

16. $x^2 - y^2 = 10,$
$2x^2 + 5xy + 3y^2 = 14.$

17. $3x^2 - 3y^2 = 9,$
$2x^2 - xy - 6y^2 = 4.$

18. $x^2 - 7x + y^2 = 7y,$
$3(x + y) = -2xy.$

PROBLEMS

148. 1. Prices of two kinds of bicycles are such that 7 of one kind and 12 of the other kind can be obtained for \$640. Three more of the latter can be purchased for \$180 than can be purchased of the former for \$120. Find the price of each.

2. A person lends \$2500 in two separate sums at the same rate of interest. At the end of one year the first sum with interest amounts to \$997.50; at the end of two years the second sum with (simple) interest amounts to \$1705. Find the two separate sums and the rate of interest.

3. Prove that if the sum of two real numbers is multiplied by the sum of their reciprocals, the product cannot be less than 4.

Let x and y be the two real numbers, p the product considered; solve for the unknown ratio $\frac{x}{y}$.

4. A traveler starts from A toward B, another traveler starts at the same time from B toward A. In two hours they meet 20 miles from A. When the second traveler arrives at A, the first is still $13\frac{5}{7}$ miles from B. Find the distance between A and B.

5. If in $ax^2 + bx + c = 0$, the coefficients are related to each other in such a way that $a + b = 2$ and $a = 2c$, what must be the value of a, so that 8 will be a root of the given quadratic equation?

6. The difference between two numbers is 12, and the difference between their cubes is 7488. What are the two numbers?

7. A sum of money at simple interest for four years amounts to \$2240. Had the rate of interest been 1 % higher, the sum would have amounted to \$80 less than this in two years. Find the capital and the rate.

8. The sum of the areas of two circles is 694.2936 sq. in.; the sum of their radii is 21. Find their radii.

9. A circular track is constructed so that the width of the track is $\frac{1}{60}$ of the inside diameter. The area of the track is 2500 sq. yd. What are the inside and outside lengths of the track?

10. Find two numbers such that their sum is equal to their product and also to the difference of their squares.

11. Find two numbers such that the sum of the numbers is equal to the sum of their squares and also to twice their product.

12. Find two numbers whose sum is 8 and whose product is 25.

13. Find three numbers such that the sum of the squares of the first two is equal to three times the square of the first minus the square of the second, and is also equal to the first minus the second plus twice the third, and also to the sum of the three numbers.

CHAPTER VII

EXPONENTS, RADICALS, IMAGINARIES

MEANINGS OF DIFFERENT KINDS OF EXPONENTS

149. The *different kinds of exponents* which have been studied thus far have been interpreted in the following manner, m and n being taken to be positive integers:

$$a^n = a \cdot a \cdot a \cdots \text{ (to } n \text{ factors)},$$

$$a^{-n} = \frac{1}{a^{+n}}, \text{ or more generally, } a^{-\frac{m}{n}} = \frac{1}{a^{+\frac{m}{n}}}, \text{ where } a \neq 0,$$

$$a^0 = 1, \text{ where } a \neq 0,$$

$$a^{\frac{m}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m.$$

While a number a has in general n different nth roots, it is agreed that only one of them shall be represented by $a^{\frac{1}{n}}$ or $\sqrt{a}$, namely, the so-called *principal root.* This restriction was made in order to avoid unnecessary complication and confusion in the interpretation of expressions and equations involving radicals. Accordingly, $\sqrt{4} = +2$, $-\sqrt{4} = -2$, $\sqrt[3]{8} = +2$, $-\sqrt[3]{8} = -2$, $\sqrt[3]{-8} = -2$, $-\sqrt[3]{-8} = +2$, $\sqrt[4]{16} = +2$, etc.

The exponents considered above are all rational numbers. But other exponents have been brought to our attention. In the study of logarithms, mention was made of the fact that logarithms are often irrational numbers. Since logarithms are really exponents, it follows that exponents may be irrational. In more advanced books still another kind of exponent is considered, namely, the exponent that is an imaginary number.*

* For a fuller treatment of the theory of exponents, consult H. B. Fine, *College Algebra*, 1904, p. 376.

OPERATIONS WITH EXPONENTS

150. Operations involving exponents are subject to the following laws:

1. Law of multiplication: $a^m \cdot a^n = a^{m+n}$.
2. Law of division: $a^m \div a^n = a^{m-n}$.
3. Law of involution: $(a^m)^n = a^{mn}$.
4. Law of evolution: $(a^m)^{\frac{1}{n}} = a^{\frac{m}{n}}$.

DIFFERENT KINDS OF NUMBERS

151. In the study of arithmetic and algebra several different kinds of numbers have come to our notice. These may be classified as follows:

Numbers	Real	(Classified according to signs)	Positive Negative
		(Classified according to expressibility in Hindu-Arabic numerals)	Rational Irrational, as $\sqrt{5}$, $\pi = 3.14159\cdots$
	Imaginary	Pure Imaginary, as $b\sqrt{-1}$, where $b \neq 0$	
		Complex Numbers, as $a + b\sqrt{-1}$, where $b \neq 0$	

Arithmetic deals with real numbers, all positive, but some of them irrational.

The irrational numbers of arithmetic arise in the process of finding roots.

In algebra use is made of the numbers encountered in arithmetic, but convenience forces upon us the need of considering also negative numbers for the purpose of indicating relations of opposition, as temperatures above or below a certain fixed point, debts or assets, distances to the right or left, etc. In the solution of quadratic equations we meet a still different type. If we try to solve $x^2 + 1 = 0$, we are confronted with the symbol $\sqrt{-1}$.

This result greatly embarrassed mathematicians of the eighteenth and of earlier centuries; no satisfactory explanation of it could be made at first. But now the symbols are recognized as constituting a new type of number, the so-called *imaginary number*, which deserves a legitimate place in algebra and is of great service in certain advanced developments of algebra, that are useful in the study of polyphase electric currents and of other advanced topics in mathematical physics. Just as negative numbers have been found truly useful in elementary algebra, so imaginary numbers have been found useful in advanced algebra. The symbol $\sqrt{-1}$, or this multiplied by any real number b, such as $b\sqrt{-1}$, is called a *pure imaginary number.*

Expressions of the type $a \pm b\sqrt{-1}$, which are the sum or difference of a number a and of a pure imaginary $b\sqrt{-1}$ (a and b being any real numbers, but $b \neq 0$), are called *complex numbers.* The term "complex" was introduced because the parts of the expression are partly real and partly imaginary.

SIMPLIFYING RADICALS

152. A *radical* is said to be in its simplest form:

(*a*) When the index of the root is as small as possible,

(*b*) When the expression under the radical sign, called the *radicand*, is integral,

(*c*) When the radicand contains no factor with a negative exponent, or raised to a power equal to or greater than the index of the root.

$\sqrt[4]{25}$ is not in its simplest form, because the index 4 of the root is not as small as possible; we have $\sqrt[4]{25} = 5^{\frac{2}{4}} = 5^{\frac{1}{2}} = \sqrt{5}$.

$\sqrt{\frac{1}{3}}$ is not in its simplest form, because the radicand, $\frac{1}{3}$, is fractional.

$\sqrt[3]{a^4b}$ is not in its simplest form, because the exponent 4 is greater than 3, the index of the root; we have $\sqrt[3]{a^4b} = a\sqrt[3]{ab}$.

$\sqrt[3]{3\,ab^2}$ is in its simplest form, since it fulfills all three conditions.

1. Simplify $\sqrt{\dfrac{4\,a^6b^7}{3}}$.

$$\sqrt{\frac{4\,a^6b^7}{3}} = \frac{2^{\frac{2}{2}}\,a^{\frac{6}{2}}b^3b^{\frac{1}{2}}}{3^{\frac{1}{2}}} \cdot \frac{3^{\frac{1}{2}}}{3^{\frac{1}{2}}} = \frac{2\,a^3b^3}{3} \cdot 3^{\frac{1}{2}}\,b^{\frac{1}{2}} = \frac{2\,a^3b^3}{3}\sqrt{3\,b}.$$

Notice that the denominator was rationalized by multiplying numerator and denominator by $3^{\frac{1}{2}}$.

2. Simplify $\sqrt[3]{3\,a^2b^{-2}c}$.

$$\sqrt[3]{3\,a^2b^{-2}c} = 3^{\frac{1}{3}}\,a^{\frac{2}{3}}b^{-\frac{2}{3}}c^{\frac{1}{3}} = \frac{3^{\frac{1}{3}}\,a^{\frac{2}{3}}b^{-\frac{2}{3}}b^{\frac{3}{3}}c^{\frac{1}{3}}}{b^{\frac{3}{3}}} = \frac{3^{\frac{1}{3}}\,a^{\frac{2}{3}}b^{\frac{1}{3}}c^{\frac{1}{3}}}{b} = \frac{1}{b}\sqrt[3]{3\,a^2bc}.$$

3. Other illustrations of simplifying are:

$$\sqrt[3]{\frac{5}{16}} = \frac{5^{\frac{1}{3}}}{2^{\frac{4}{3}}} = \frac{5^{\frac{1}{3}} \cdot 2^{\frac{2}{3}}}{2^{\frac{4}{3}} \cdot 2^{\frac{2}{3}}} = \frac{1}{4}\sqrt[3]{20}.$$

$$\sqrt[5]{\frac{27\,a^6}{b^3}} = \frac{3^{\frac{3}{5}}\,a^{\frac{6}{5}}}{b^{\frac{3}{5}}} = \frac{3^{\frac{3}{5}} \cdot a \cdot a^{\frac{1}{5}}b^{\frac{2}{5}}}{b^{\frac{3}{5}} \cdot b^{\frac{2}{5}}} = \frac{a}{b}\sqrt[5]{27\,ab^2}.$$

$$\sqrt{72 - 36\sqrt{3}} = 36^{\frac{1}{2}}(2 - \sqrt{3})^{\frac{1}{2}} = 6\sqrt{2 - \sqrt{3}}.$$

EXERCISES

153. Remove all negative exponents, compute the values, and simplify the expressions:

1. $27^{\frac{2}{3}}$.
2. $16^{\frac{3}{4}}$.
3. $27^{-\frac{2}{3}}$.
4. $16^{-\frac{3}{4}}$.
5. $64^{-\frac{1}{2}}$.
6. $64^{\frac{5}{6}}$.
7. $64^{-\frac{5}{6}}$.
8. $(\frac{2}{3})^{-3}$.
9. $8^{-\frac{2}{3}}$.
10. $(\frac{1}{10})^{-2}$.
11. $(.16)^{\frac{3}{2}}$.
12. $(.125)^{\frac{4}{3}}$.
13. $(-\frac{3}{4} + 5)^0$.
14. $7^0 \times (.25)^{-\frac{3}{2}}$.
15. $(.49)^{1.5}$.
16. $(100^0 + 10)^{-1}$.
17. $\dfrac{5^{-2} - 25^{-1}}{4^{\frac{3}{4}}}$.
18. $\dfrac{5^n \cdot 5 - 5^{n-1}}{5^{n-2}}$.
19. $\dfrac{a^{m-n} + a^{-m}b}{a^{n-m}}$.
20. $\dfrac{m^{-2} - n^{-2}}{m^{-1} - n^{-1}}$.
21. $\dfrac{a^{-4} + a^{-2} + 1}{a^{-2} - a^{-1} + 1}$.

22. $\sqrt{60}$. 25. $\sqrt{2000}$. 28. $\sqrt[3]{81}$. 31. $\sqrt[3]{-800}$.

23. $\sqrt{200}$. 26. $7\sqrt{20}$. 29. $\sqrt[3]{48}$. 32. $\sqrt[5]{729}$.

24. $\sqrt{98}$. 27. $12\sqrt{8}$. 30. $\sqrt[3]{08}$. 33. $\sqrt[5]{2430}$.

34. $2\sqrt{6} \times 5\sqrt{3}$. 38. $3\sqrt{10} \times 24\sqrt{6} \div \sqrt{15}$.

35. $5\sqrt{2} \times 7\sqrt{6}$. 39. $3\sqrt{10} \times 2\sqrt{15} \div 3\sqrt{2}$.

36. $2\sqrt{15} \times 5\sqrt{2} \times 2\sqrt{3}$. 40. $5\sqrt{15} \times 3\sqrt{6} \div 2\sqrt{5}$.

37. $2\sqrt{15} \div 5\sqrt{3} \times 2\sqrt{2}$. 41. $(6 \cdot 25)^{-\frac{1}{2}}$.

42. $(2.25)^{\frac{3}{2}}$. 46. $(.216)^{\frac{4}{3}}$. 50. $(\frac{81}{625})^{\frac{3}{4}}$.

43. $(.25)^{-\frac{3}{2}}$. 47. $(.0625)^{-\frac{3}{2}}$. 51. $-(6.25)^{\frac{1}{2}}$.

44. $(.125)^{-\frac{2}{3}}$. 48. $(-.125)^{\frac{4}{3}}$. 52. $(a^{\frac{1}{2}}b^{\frac{1}{3}})^6$.

45. $(-.008)^{-\frac{4}{3}}$. 49. $(\frac{36}{49})^{\frac{3}{2}}$. 53. $(a^2b^3)^{-\frac{1}{4}}$.

54. $(a^{-m}b^{-n})^{-\frac{1}{mn}}$. 56. $c^{\frac{2}{3}} \div c^{\frac{4}{5}} \times c^{\frac{6}{7}}$.

55. $a^{\frac{2}{3}} \div a^{\frac{3}{4}} \times a^{\frac{4}{5}}$. 57. $2^{\frac{1}{2}} \times 64^{\frac{1}{4}} \div 729^{\frac{1}{6}}$.

Rationalize the denominators of the following fractions and then find their values to three figures:

58. $\dfrac{1}{\sqrt{2}}$. Hint. $\dfrac{1}{\sqrt{2}} = \dfrac{1}{\sqrt{2}} \cdot \dfrac{\sqrt{2}}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$. 59. $\dfrac{1}{2\sqrt{3}}$.

60. $\dfrac{8}{\sqrt{14}}$. 61. $\dfrac{1}{\sqrt{2.3}}$. 62. $\sqrt{\frac{5}{7}}$. 63. $\dfrac{\sqrt{1.3}}{\sqrt{.22}}$.

Calculate the values of the following expressions when $m = \sqrt{2}$, $n = \sqrt{3}$, $p = \sqrt{5}$, using the tables in § 197:

64. $7m^2 - n^2$. 67. $p^2n^3 - n^2m^3$. 70. $(n + p)m$.

65. $\frac{1}{5}p^2 - m^4n^4$. 68. $\dfrac{m^2 + n^2}{2p}$. 71. $\dfrac{p^3 - n^3}{pn}$.

66. $m^2n^3 + n^2p^3$. 69. $\dfrac{m - n}{5p^2}$. 72. $\dfrac{m^4 - n^4}{-mnp}$.

73. Which of the expressions is the larger, $\sqrt{5}$ or $\sqrt[3]{11}$?

$$\sqrt{5} = 5^{\frac{1}{2}} = 5^{\frac{3}{6}} = 125^{\frac{1}{6}}.$$
$$\sqrt[3]{11} = 11^{\frac{1}{3}} = 11^{\frac{2}{6}} = 121^{\frac{1}{6}}.$$

But $125^{\frac{1}{6}} > 121^{\frac{1}{6}}$. Hence $\sqrt{5}$ is the larger.

Arrange in the order of magnitude in each example:

74. $\sqrt[3]{11}$, $\sqrt[4]{21}$.

75. $\sqrt{2}$, $\sqrt[3]{2.9}$.

76. $\sqrt{4.4}$, $\sqrt[3]{9}$.

77. 2, $\sqrt{4.1}$, $\sqrt[3]{8.3}$.

78. $\sqrt[4]{2}$, $\sqrt[3]{1.7}$, $\sqrt{1.4}$.

79. $\sqrt[4]{3}$, $\sqrt[3]{2.3}$, $\sqrt{1.7}$.

80. Using logarithms, compute the value of $\sqrt[3]{7943}$.

$$\text{Log } 7943 = 3.9000.$$
$$\text{Log } \sqrt[3]{7943} = 1.3000.$$
$$\sqrt[3]{7943} = 19.95, \text{ nearly.}$$

81. Using logarithms, compute the value of $x = (59.45)^{-\frac{2}{3}}$.

$$(59.45)^{-\frac{2}{3}} = \frac{1}{(59.45)^{\frac{2}{3}}}.$$

$$\text{Log } 1 = 0.0000$$
$$\text{Log } 59.45 = 1.7742.$$
$$\text{Log } (59.45)^{\frac{2}{3}} = \tfrac{2}{3}(1.7742) \qquad = \underline{1.1828}$$
$$\text{Log } x = \bar{2}.8172$$
$$x = .06564, \text{ nearly.}$$

Compute the values of the following radicals:

82. $\sqrt[4]{908}$.

83. $\sqrt[5]{1083}$.

84. $\sqrt[6]{790}$.

85. $(12)^{\frac{1}{10}}$.

86. $(154)^{-\frac{1}{2}}$.

87. $(376)^{\frac{3}{4}}$.

ADDITION AND SUBTRACTION OF RADICALS

154. The indicated additions and subtractions of radicals can be reduced to one term, only when the terms in the expression are similar after they are simplified.

Thus,
$$\sqrt{18} + \sqrt{50} = 3\sqrt{2} + 5\sqrt{2} = 8\sqrt{2},$$
$$\sqrt[6]{64\,a^3} + \sqrt[4]{81\,a^2} = 2\,a^{\frac{1}{2}} + 3\,a^{\frac{1}{2}} = 5\,a^{\frac{1}{2}} = 5\sqrt{a}.$$

EXERCISES

155. Perform the indicated operations:

1. $\sqrt{63}+\sqrt[4]{784}$.
2. $\sqrt{294}+\sqrt[4]{576}$.
3. $\sqrt[3]{24}+\sqrt[3]{81}+\sqrt[3]{375}$.
4. $50\sqrt[3]{\frac{7}{1000}}+\sqrt[3]{7000}-\sqrt[3]{56}$.
5. $\sqrt{4\,a^3c^2}-\sqrt{9\,a^5b}-\sqrt{16\,a^7d^2}$.
6. $6\sqrt{3}+8\sqrt{\frac{3}{4}}+\sqrt{\frac{27}{4}}$.
7. $4\sqrt{\frac{15}{2}}+15\sqrt{\frac{10}{3}}-60\sqrt{\frac{6}{5}}$.
8. $\sqrt[3]{a^4}-\sqrt[6]{a^2b^6}+\sqrt[9]{a^3c^9}$.
9. $ab\sqrt{\frac{5\,a}{b}}+2\,ab\sqrt{\frac{5\,b}{a}}+5\sqrt{\frac{ab}{5}}$.
10. $n^3\sqrt{\frac{m}{n^3}}-n^4\sqrt{\frac{m}{n^5}}+m^2n^2\sqrt{\frac{n^3}{m}}$.

MULTIPLICATION OF RADICALS

156. Real radicals of the same index are multiplied together according to the formula

$$a\sqrt[n]{b}\times c\sqrt[n]{d}=ac\sqrt[n]{bd}.$$

When the real radicals have different indices, it is easiest to change to the exponential notation, then perform the multiplication and, after reducing fractional exponents to a common denominator, return to the radical notation. This may be shown as follows:

$$a\sqrt[n]{b}\times c\sqrt[m]{d}=ab^{\frac{1}{n}}\times cd^{\frac{1}{m}}=acb^{\frac{m}{mn}}d^{\frac{n}{mn}}=ac\sqrt[mn]{b^md^n}.$$

Special care must be exercised when the expressions to be multiplied involve imaginary numbers. It is customary, for brevity, to represent the pure imaginary $\sqrt{-1}$ by the single

letter i. Hence the complex number $a + \sqrt{-1}\,b$ may be written $a + ib$.

The following has been adopted as a definition of $\sqrt{-1}$ or i:

The number $\sqrt{-1}$ is one whose square is -1.

Accordingly, $$(\sqrt{-1})^2 = i^2 = -1.$$

All multiplications involving imaginary numbers must be made to conform with this fundamental definition.

Particular care must be exercised to avoid the following procedure:

$$\sqrt{-1} \text{ times } \sqrt{-1} = \sqrt{(-1)(-1)} = \sqrt{+1} = 1.$$

This procedure is in violation of the definition of $\sqrt{-1}$ given above, and must therefore be avoided. Always take $\sqrt{-1} \times \sqrt{-1} = -1$, or $i \cdot i = -1$.

The multiplication of expressions involving imaginaries *can be performed easiest, if we take the precaution to write a radical $\sqrt{-a}$ in the form $\sqrt{-1}\sqrt{a}$ or $i\sqrt{a}$.*

For example,

$$\sqrt{-a} \times \sqrt{-b} = i\sqrt{a} \times i\sqrt{b} = i^2\sqrt{ab} = -\sqrt{ab}.$$

1. Multiply $2\sqrt{a} - 3\sqrt{2b} + 4\sqrt{3ab}$ by $5\sqrt{ab}$.

$$\begin{array}{l} 2\sqrt{a} - 3\sqrt{2b} + 4\sqrt{3ab} \\ \underline{5\sqrt{ab}} \\ 10a\sqrt{b} - 15b\sqrt{2a} + 20ab\sqrt{3} \end{array}$$

2. Multiply $\sqrt{a} + 2\sqrt{-b} - 3c$ by $\sqrt{-ab}$.

$$\begin{array}{l} \sqrt{a} + 2i\sqrt{b} - 3c \\ \underline{i\sqrt{ab}} \\ ai\sqrt{b} - 2b\sqrt{a} - 3ci\sqrt{ab} \end{array}$$

EXERCISES

157. Perform the indicated operations and simplify the result:

1. $\sqrt{5} \cdot \sqrt{125}$.
2. $\sqrt[3]{3} \cdot \sqrt[3]{9}$.
3. $3\sqrt{a} \cdot 5\sqrt{ab}$.
4. $\sqrt[3]{a^2} \div \sqrt[4]{a^3} \times \sqrt[5]{a^4}$.
5. $\sqrt[3]{a^4} \div \sqrt{a} \times \sqrt[4]{a^3}$.
6. $a\sqrt{-1} \times 26\sqrt{-1}$.
7. $3i \cdot 4i\sqrt{5}$.
8. $4i\sqrt{c} \div 2i$.
9. $5\sqrt{-1}\sqrt{3} \times 6\sqrt{-7}$.
10. $\sqrt[3]{12} \times \sqrt{\frac{1}{2}}$.
11. $(\sqrt[3]{a - b\sqrt{c}})^{\frac{1}{3}}$.
12. $-\sqrt[4]{2a^3} \times \sqrt{2ab}$.
13. $\sqrt{2-x} \cdot \sqrt{2+x}$.
14. $\sqrt[3]{a-b} \cdot \sqrt{a+b}$.
15. $(\sqrt{2a} + \sqrt{3b})^2$.
16. $(a + \sqrt{b})^3$.
17. $(a + \sqrt{-b})^3$.
18. $\sqrt{ab} \div \sqrt{a}$.
19. $\sqrt[3]{a^2b} \div \sqrt{ab}$.
20. $3\sqrt{x-5} \cdot 4\sqrt{2x-2}$.
21. $\sqrt{-ab} \div \sqrt{-a}$.
22. $ia \times ib\sqrt{2} \times ic$.
23. $\sqrt{-3} \cdot \sqrt{-5} \cdot \sqrt{-\frac{1}{12}}$.
24. $i\sqrt{x} \cdot i\sqrt{xy} \cdot i\sqrt{yz}$.
25. $\sqrt[3]{-5} \times \sqrt[3]{-7} \times \sqrt[3]{-\frac{1}{25}}$.
26. $(2\sqrt{x} - 3\sqrt{y} + \sqrt{2z}) \cdot 5\sqrt{xyz}$.
27. $6\sqrt[3]{xy}(2\sqrt{x} + \sqrt{y} - z)$.
28. $(2\sqrt{2} - 3\sqrt{-2} + \sqrt[3]{-2} - 2\sqrt[3]{2})\sqrt[3]{2}$.
29. $(2\sqrt{5} - \sqrt[3]{6} - 2\sqrt{-1})^2$.
30. $(\sqrt{3} - 3\sqrt{-2})^3$.

31. Change to an equivalent fraction having a rational denominator, $\dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$.

If $\sqrt{3} - \sqrt{2}$ is multiplied by $\sqrt{3} + \sqrt{2}$, the product is $(\sqrt{3})^2 - (\sqrt{2})^2$ or $3 - 2$. Hence, multiply both numerator and denominator of the given fraction by $\sqrt{3} + \sqrt{2}$. We obtain

$$\frac{(\sqrt{3}+\sqrt{2})^2}{(\sqrt{3})^2 - (\sqrt{2})^2} = \frac{3 + 2\sqrt{6} + 2}{3 - 2} = \frac{5 + 2\sqrt{6}}{1} = 5 + 2\sqrt{6}.$$

Change to equivalent expressions having a rational denominator:

32. $\dfrac{3}{\sqrt{3}-\sqrt{5}}$. 34. $\dfrac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}$. 36. $\dfrac{4+\sqrt{5}}{2+\sqrt{3}}$.

33. $\dfrac{1}{\sqrt{6}+\sqrt{5}}$. 35. $\dfrac{\sqrt{2}+\sqrt{b}}{\sqrt{2}-\sqrt{b}}$. 37. $\dfrac{3}{2\sqrt{5}-3\sqrt{3}}$.

38. Is $-1+\sqrt{5}$ a root of the equation $x^2+2x-4=0$?

Write $-1+\sqrt{5}$ for x,

$(-1+\sqrt{5})^2+2(-1+\sqrt{5})-4=1-2\sqrt{5}+5-2+2\sqrt{5}-4=0.$

Hence, $-1+\sqrt{5}$ is a root of the equation.

Is the binomial written after each equation a root of that equation?

39. $x^2-4x-1=0$, $2+\sqrt{3}$.

40. $x^2-x-3=0$, $\frac{1}{2}(1-\sqrt{13})$.

41. $x^2+3x+1=0$, $3+\sqrt{5}$.

42. $2x^2-x-2=0$, $\frac{1}{4}(1-\sqrt{17})$.

THE SQUARE ROOT OF $a \pm 2\sqrt{b}$.

158. Since $(\sqrt{3}\pm\sqrt{2})^2=3\pm2\sqrt{6}+2=5\pm2\sqrt{6}$, it follows that $\sqrt{5\pm2\sqrt{6}}=\sqrt{3}\pm\sqrt{2}$.

We notice that 5 is the sum of 2 and 3; 6 their product. This suggests the following rule for finding the square root of $a+2\sqrt{b}$:

Find two numbers whose sum is a and whose product is b.

Write down the square root of the larger number, plus (or minus) the square root of the smaller.

This rule is of practical use only when the two numbers sought are rational. In other cases the resulting expression is more complicated than the given expression; hence to be rejected.

1. Find the square root of $8 + \sqrt{60}$.

Writing the expression in the form $a \pm 2\sqrt{b}$, $8 + 2\sqrt{15}$.
The two numbers whose sum is 8 and product 15 are 5 and 3.
Hence, $\sqrt{8 + \sqrt{60}} = \sqrt{5} + \sqrt{3}$.

2. Find $\sqrt{6 - 4\sqrt{2}}$.

Changing to the form $a \pm 2\sqrt{b}$, $6 - 2\sqrt{8}$.
The two numbers whose sum is 6 and product 8 are 4 and 2.
Hence, $\sqrt{6 - 4\sqrt{2}} = \sqrt{4} - \sqrt{2} = 2 - \sqrt{2}$.

EXERCISES

159. Find the square root of:

1. $3 + 2\sqrt{2}$.
2. $7 - 4\sqrt{3}$.
3. $9 + 2\sqrt{14}$.
4. $9 + 2\sqrt{20}$.
5. $15 + 6\sqrt{6}$.
6. $8 + 2\sqrt{7}$.
7. $9 + 6\sqrt{2}$.
8. $\frac{9}{4} - \sqrt{2}$.
9. $\frac{7}{12} - \sqrt{\frac{1}{3}}$.
10. $x - \sqrt{x^2 - 1}$.
11. $(a^2 + 4b) - 4a\sqrt{b}$.
12. $x + 2y + 2\sqrt{2xy}$.

IRRATIONAL EQUATIONS

160. An *irrational* or *radical equation* is an equation in which the unknown number x and expressions containing it occur under radical signs or with fractional exponents.

For example, $2 - \sqrt{x} = \sqrt{x - 1}$, $x^{\frac{2}{3}} + 5x^{\frac{1}{3}} = 6$.

It is agreed among mathematicians that in equations of this sort, only the principal roots of $\sqrt{x}$, $\sqrt{x - 1}$, $x^{\frac{1}{3}}$ are to be taken.

The solution of irrational equations may be explained by examples.

1. Solve $2 - \sqrt{x} - \sqrt{x-1} = 0$.

Solution. Transpose one of the radicals, $2 - \sqrt{x} = \sqrt{x-1}$.

Square both sides, $4 - 4\sqrt{x} + x = x - 1$.

Simplify and transpose to the right all terms except $4\sqrt{x}$, $4\sqrt{x} = 5$.

Square both sides again and solve for x, $x = \frac{25}{16}$.

Substitute in the given equation, $2 - \sqrt{\frac{25}{16}} = \sqrt{\frac{25}{16} - 1}$,

$$2 - \tfrac{5}{4} = \tfrac{3}{4},$$

$$\tfrac{3}{4} = \tfrac{3}{4}.$$

Hence $x = \frac{25}{16}$ is the value sought.

2. Solve $1 + \sqrt{x+3} = \sqrt{x}$.

Square both sides, $1 + 2\sqrt{x+3} + x + 3 = x$. (1)

Simplify and isolate the radical, $\sqrt{x+3} = -2$. (2)

Square both sides again, $x + 3 = 4$, (3)

$$x = 1.$$

Substitute x in the given equation, $1 + \sqrt{4} = \sqrt{1}$,

$$1 + 2 = 1.$$

This is absurd; $x = 1$ is not a root of the given equation. This is an example of a radical equation which cannot be satisfied by *any* value of x. *The equation is impossible.* It is therefore not true of radical equations that values of x satisfying them always exist.

It will be instructive to examine just how the false value $x = 1$ was obtained. When both sides of an equation involving x are squared, new values of x may be introduced, giving rise to *extraneous roots.* By substitution we can ascertain in which place the extraneous root came in. It is found that $x = 1$ does not satisfy equations (1) and (2), but it does satisfy 3. Hence, the extraneous root $x = 1$ was introduced when both sides of the equation $\sqrt{x+3} = -2$ were squared.

Extraneous roots arise here in the same way as in the solution of fractional equations. One simple way of determining whether a value of x obtained by squaring both sides is an extraneous root or not, is by substitution in the original equation; if it is not satisfied, then the root is extraneous.

From the explanation given above it is evident that, though all operations in finding x may be performed without error, the answer may nevertheless be false. *It is therefore necessary to substitute the value of x in the original irrational equation, to ascertain whether that value is true or false.*

While $x = 1$ is not a root of $1 + \sqrt{x+3} = \sqrt{x}$, it will be noticed that it *is* a root of $1 - \sqrt{x+3} = -\sqrt{x}$. The solution of the latter equation gives rise to equation (3), and to $x = 1$.

3. Solve $\sqrt{x-2} + \sqrt{3x+3} = 3.$ (1)

Transpose so that the most complicated radical stands alone on one side of the equation,

$$\sqrt{x-2} - 3 = -\sqrt{3x+3}. \quad (2)$$

Square both sides, $x - 2 - 6\sqrt{x-2} + 9 = 3x + 3.$ (3)

Simplify, keep only the radical on left side, $-6\sqrt{x-2} = 2x - 4.$ (4)

Divide both sides by 2 $-3\sqrt{x-2} = x - 2.$ (5)

Square both sides, $9x - 18 = x^2 - 4x + 4.$ (6)

Solve the quadratic $x^2 - 13x + 22 = 0.$

$$(x-2)(x-11)$$

$$x = 2,\ x = 11.$$

Substitute $x = 2$ in (1), $\sqrt{2-2} + \sqrt{9} = 3.$

$$3 = 3.$$

Hence $x = 2$ is a root of (1).

Substitute $x = 11$ in (1), $\sqrt{11-2} + \sqrt{33+3} = 3.$

$$3 + 6 = 3.$$

Hence $x = 11$ is not a root of (1). Nor is it a root of (2), (3), (4), and (5). But it *is* a root of (6). It follows that $x = 11$ is an extraneous root, obtained in squaring (5).

4. Solve $\sqrt{6-x} + \sqrt{x+7} = 5.$

Place one radical on one side by itself, $\sqrt{6-x} = 5 - \sqrt{x+7}.$

Square both sides, $6 - x = 25 - 10\sqrt{x+7} + x + 7.$

Simplify and keep only the radical on left side,

$$10\sqrt{x+7} = 2x + 26.$$

Divide both sides by 2, $5\sqrt{x+7} = x + 13.$

Square both sides, $25x + 175 = x^2 + 26x + 169.$

Solve the quadratic $x^2 + x - 6 = 0.$

$$(x+3)(x-2) = 0.$$

$$x = -3,\ x = 2.$$

Substitute $x = -3$ in the given equation,

$$\sqrt{9} + \sqrt{4} = 5,$$
$$3 + 2 = 5.$$

Hence $x = -3$ is a root of the given equation.

Substitute $x = 2$ in the given equation,

$$\sqrt{4} + \sqrt{9} = 5,$$
$$2 + 3 = 5.$$

Hence $x = 2$ is also a root of the given equation.

EXERCISES

161. Solve the following irrational equations, testing carefully each value of x that may arise:

1. $\sqrt{x+4} = 4 - \sqrt{x-4}.$
2. $\sqrt{x-2} + \sqrt{x+3} = 5.$
3. $\sqrt{x-1} + \sqrt{x+6} = \sqrt{4x+9}.$
4. $\sqrt{x} + \sqrt{x+1} = \sqrt{2x+1}.$
5. $\sqrt{y-2} = \sqrt{y} + \sqrt{2}.$
6. $\sqrt{y+1} - \sqrt[3]{y+1} = 0.$
7. $4\sqrt{5x^2 - 5x + 9} + 3 = 31.$
8. $\sqrt{x^2 + 21} + x = 21.$

IRRATIONAL EQUATIONS QUADRATIC IN FORM

162. The equation $x^{\frac{2}{3}} + 5x^{\frac{1}{3}} + 6 = 0$ is said to be *quadratic in form*, because it contains only two different powers of x, one exponent of x being double the other exponent.

If we put $x^{\frac{1}{3}} = y$, then $x^{\frac{2}{3}} = y^2$ and we obtain the quadratic

$$y^2 + 5y + 6 = 0.$$

Solving this we obtain $\quad y = -2$ and $y = -3.$

Hence, $\quad x^{\frac{1}{3}} = -2$ and $x^{\frac{1}{3}} = -3.$

Cubing both members, $\quad x = -8$ and $x = -27.$

Substituting $x = -8$ in the given equation,

$$(-8)^{\frac{2}{3}} + 5(-8)^{\frac{1}{3}} + 6 = 0.$$

Simplify,

$$(-2)^2 + 5(-2) + 6 = 0,$$
$$4 - 10 + 6 = 0.$$

Hence, $x = -8$ is a root of the given equation.

Substituting $x = -27$ in the given equation.

$$(-27)^{\frac{2}{3}} + 5(-27)^{\frac{1}{3}} + 6 = 0,$$
$$(-3)^2 + 5(-3) + 6 = 0,$$
$$9 - 15 + 6 = 0.$$

Hence, $x = -27$ is a root of the given equation.

Solve $x^{\frac{3}{2}} - 7x^{\frac{3}{4}} = 8$.

The exponent $\frac{3}{2}$ is double the exponent $\frac{3}{4}$; hence the equation is quadratic in form.

Let $\quad x^{\frac{3}{4}} = y$, then $x^{\frac{3}{2}} = y^2$.

Solving,

$$y^2 - 7y = 8,$$
$$y = 8,\ y = -1.$$

We obtain $\quad x^{\frac{3}{4}} = 8$ and $x^{\frac{3}{4}} = -1. \qquad (1)$

The exponent $\frac{3}{4}$ in $x^{\frac{3}{4}}$ signifies the third power of the fourth root of x; we wish to find x itself.

To find the value of $x^{\frac{1}{4}}$, we must extract the *cube root* of both sides of (1).

We obtain $\quad x^{\frac{1}{4}} = +2$ and $x^{\frac{1}{4}} = -1. \qquad (2)$

To find the value of x, we must raise both sides of (2) to the *fourth power*.

We obtain $\quad x = 16$ and $x = 1$.

Substitute $x = 16$ in the given equation,

$$(16)^{\frac{3}{2}} - 7(16)^{\frac{3}{4}} = 8,$$
$$(4)^3 - 7(2)^3 = 8,$$
$$64 - 56 = 8.$$

Hence, $x = 16$ is a root of the given equation.

Substitute $x = 1$ in the given equation,

$$(1)^{\frac{3}{2}} - 7(1)^{\frac{3}{4}} = 8,$$
$$1 - 7 = 8.$$

Hence, $x = 1$ is *not* a root of a given equation. Where did this false value enter? We see that $x = 1$ satisfies neither $x^{\frac{3}{4}} = -1$ nor $x^{\frac{1}{4}} = -1$; hence this extraneous root appeared when the sides of $x^{\frac{1}{4}} = -1$ were raised to the fourth power.

In solving (1), we might have reversed the two operations, by first raising both sides to the fourth power and then extracting the cube root. But the operations would have involved larger numbers in the case of $x^{\frac{3}{4}} = 8$.

Be careful to avoid mistakes in solving equations like $x^{\frac{3}{4}} = 8$.

A frequent error is to conclude from $x^{\frac{3}{4}} = 8$

that $x = 8^{\frac{3}{4}}$.

As a matter of fact, $x = 8^{\frac{4}{3}}$.

The safer way is to proceed by two steps as is done above:

The first step (extraction of the cube root) changes the exponent of x from $\frac{3}{4}$ to $\frac{1}{4}$.

The second step (raising to the fourth power) changes the exponent of x from $\frac{1}{4}$ to 1.

EXERCISES

163. Solve:

1. $x^{\frac{2}{3}} - 7x^{\frac{1}{3}} + 10 = 0.$
2. $8x^{\frac{3}{2}} - 65x^{\frac{3}{4}} + 8 = 0.$
3. $z^{\frac{2}{5}} - 3z^{\frac{1}{5}} + 2 = 0.$
4. $(x+1)^{\frac{2}{3}} - 11(x+1)^{\frac{1}{3}} + 30 = 0.$
5. $x^{\frac{1}{4}} - x^{\frac{1}{2}} + 132 = 0.$
6. $y^{\frac{3}{5}} + 19y^{\frac{3}{10}} - 216 = 0.$
7. $(x+1)^3 - 60(x+1)^{\frac{3}{2}} = 256.$
8. $x^{\frac{4}{3}} + 6x^{\frac{2}{3}} - 55 = 0.$

GRAPHIC REPRESENTATION OF COMPLEX NUMBERS

164. The graphic representation of ordinary positive and negative numbers is familiar to all students of algebra. On a straight line, usually drawn horizontally, a point is chosen as the starting point or *origin*; positive numbers are shown by distances to the right, negative numbers by distances to the left.

The graphic representation of complex numbers like $a + ib$, where $i = \sqrt{-1}$, is not so obvious. In the seventeenth century John Wallis of

the University of Oxford made attempts to find a geometrical interpretation of such numbers, but was not able to devise a general and consistent scheme. Not till the close of the eighteenth century did a satisfactory plan suggest itself to mathematicians. In 1797 a Norwegian surveyor, by the name of Caspar Wessel, found a graphic representation which agrees with that now adopted. But his writings failed to attract the notice of mathematicians. A little later a Frenchman by the name of Argand had a similar experience. More successful in reaching the ear of the mathematical public on this matter was the German mathematician, Carl Friedrich Gauss. He placed the theory of complex numbers on a firm basis and, by his reputation, induced others to enter upon a more careful study of this subject and to adopt the geometrical interpretation that is given now in works on algebra.

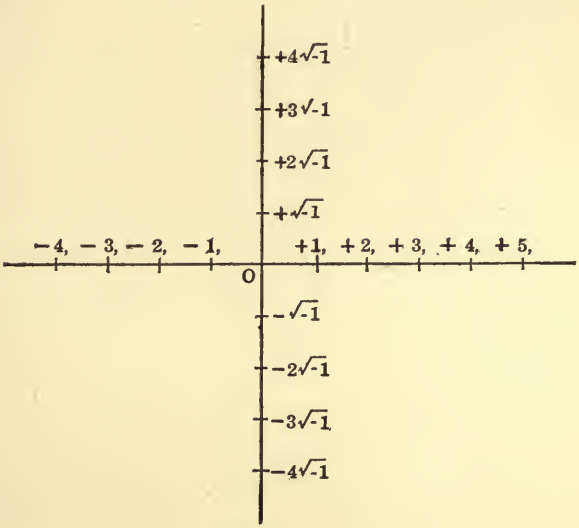

FIG. 23.

165. This interpretation of imaginary or complex numbers is really quite simple. Two axes are drawn perpendicular to each other, one axis horizontal, the other vertical, as in Fig. 23. Along the horizontal axis real numbers are marked off, positive

numbers to the right and negative numbers to the left of the origin. The pure imaginary numbers $+\sqrt{-1}$, $+2\sqrt{-1}$, $+3\sqrt{-1}$, ··· are marked off along the perpendicular axis, from the origin *up*. The pure imaginary numbers $-\sqrt{-1}$, $-2\sqrt{-1}$, $-3\sqrt{-1}$, ··· are marked off along this same perpendicular, but from the origin *down*.

Complex numbers are pictured in this diagram by points in the plane. In case of $3 + i\,4$, measure off three unit distances on the horizontal axis to the right, and 4 unit distances *up*. This locates the point A in the plane (Fig. 24). The point A in the plane is accordingly the graphic representation of $3 + i\,4$.

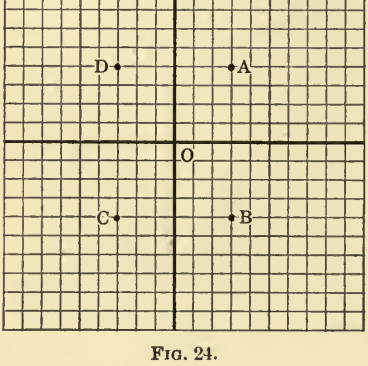

FIG. 24.

To find the geometric representation of $3 - i\,4$, pass from the origin 3 units to the right, as before, and 4 units *down;* the point B in Fig. 24 represents $3 - i\,4$.

Similarly, C represents $-3 - i\,4$, and D represents $-3 + i\,4$.

The representation is similar for other complex numbers. It is seen that every complex number can be represented in this way, and that every point in the plane stands for some number. All real numbers are confined to the horizontal axis; all pure imaginary numbers are confined to the vertical axis; all other numbers $a \pm ib$ lie in one or another of the four quadrants.

The axis for the pure imaginaries is used in much the same way as the y-axis when the graph of an equation containing the variables x and y is drawn. Only now the vertical axis carries *imaginary* numbers, while before the y-axis carried *real* values of the variable y.

EXERCISES

166. Locate the points which represent the following complex numbers:

1. $5 + i\,4$.
2. $-5 - i\,6$.
3. $-5 + i\,6$.
4. $5 - i\,6$.
5. $-1 + i$.
6. $6 + i\,6$.
7. $3 - 4\sqrt{-1}$.
8. $4 + 5\sqrt{-1}$.
9. $\sqrt{3} + \sqrt{-5}$.
10. $(2 + i\,3) + (1 + i\,2)$.

[*Hint.* $(2 + i\,3) + (1 + i\,2) = 3 + i\,5$; locate $3 + i\,5$.]

11. $(4 + i) - (2 + i\,2)$.
12. $(5 - i\,2) + (3 - i\,4)$.
13. $(6 - i\,5) - (6 - i\,7)$.

CHAPTER VIII

SERIES AND LIMITS

ARITHMETICAL SERIES

167. The numbers 5, 9, 13, 17, 21 appear upon examination to have been selected according to some law and arranged in a definite order. Each number after the first is greater than the one immediately preceding by 4. Such a regulated succession of numbers is called *series.* When, as here, the increase is the same throughout, the series receives the special name of *arithmetical series* or *arithmetical progression.*

An *arithmetical series* is a succession of numbers in which each number after the first minus the preceding one always gives the same difference.

This difference is called the *common difference.* Instead of increasing, the numbers in the series may decrease, so that the first number is the largest and the common difference is negative.

Arithmetical series are frequently encountered in the study of mathematics, hence it is desirable to develop certain formulas relating to such series.

EXERCISES

168. State which of the following series are *arithmetical series:*

1. 10, 8, 6, 4, 2.
2. $\frac{1}{2}$, 1, $1\frac{1}{2}$, 2, $2\frac{1}{2}$, 3.
3. 10, 9, 7, 6, 4.
4. $a, a+d, a+2d, a+3d.$
5. $\frac{9}{\sqrt{2}}, \frac{7}{\sqrt{2}}, \frac{5}{\sqrt{2}}, \frac{3}{\sqrt{2}}, \frac{1}{\sqrt{2}}.$
6. $x, x-y, x-2y, x-3y.$

THE LAST TERM

169. If a stands for the first number or term of an arithmetical series, and d for its common difference, then the series may be written in general terms thus,

$$a,\ a+d,\ a+2d,\ a+3d,\ a+4d, \text{ etc.}$$

Let l stand for the last term of the arithmetical series. The value of the last term evidently depends upon three things; namely, the value of a, the value of d, and also the number of terms in the series. Denote the number of terms by n.

It is to be observed that the coefficient of d in the second term is 1, in the third term is 2, in the fourth term is 3, in the fifth term is 4. If n denotes the number of terms, what must be the coefficient of d in the last term? From what we have observed it must be *one less than the number of the term*, that is, $n-1$. Hence we have the formula for the nth term,

$$l = a + (n-1)d. \qquad (A)$$

EXERCISES

170. 1. Find the 10th term in the arithmetical series, 2, 7, 12, 17, . . .

Here $a = 2$, $d = 5$, $n = 10$.

Substitute these values in (A), $\quad l = 2 + (10-1)5,$

$l = 47.$

Hence the 10th term is 47, as may be verified by writing down all the terms to the 10th term.

2. Find the 15th term of the series $-2, -4, -6, -8, \cdots$.

3. Find the 12th term of the series $1, 1\frac{1}{2}, 2, 2\frac{1}{2}, 3, \cdots$.

4. Find the 20th term of the series $5, 5+2x, 5+4x, 5+6x, \cdots$.

5. Find the 24th term of the series $\sqrt{2}, \sqrt{2}+1, \sqrt{2}+2, \sqrt{2}+3, \cdots$.

6. Find the $(n-1)$th term of the series 3, 6, 9, 12, 15, $\cdots$.

Find the expression for any given term (the nth term) of the following series of numbers:

7. 3, 5, 7, 9, $\cdots$.
8. 7, 12, 17, 22, $\cdots$.
9. 13, $17\frac{1}{2}$, 22, $26\frac{1}{2}$, $\cdots$.
10. $\frac{2}{3}$, $1\frac{1}{24}$, $1\frac{5}{12}$, $\cdots$.
11. 66, 59, 52, $\cdots$.
12. 2.4, 2.1, 1.8, $\cdots$.

13. Find the $(n-2)$th term of the series 5, 1, -3, -7, $\cdots$.

14. A bullet is fired vertically upward so that at the end of the first second it has a velocity of 200 ft. per sec., at the end of the second second a velocity of 168 ft. per sec., at the end of the third second a velocity of 136 ft. per. sec., and so on. Compute the velocity at the end of the sixth second, at the end of the tenth second. Interpret the second answer.

15. A body falling from rest falls 16 ft. during the first second, 48 ft. during the second, 80 ft. during the third, 112 ft. during the fourth, and so on. How far will it fall during the ninth second?

ARITHMETICAL MEANS

171. The *arithmetical means* between two numbers are numbers which, together with the two given numbers as first and last terms, form an arithmetical series.

If the two given numbers are 5 and 50, then 14, 23, 32, 41 are four arithmetical means, because 5, 14, 23, 32, 41, 50 is an arithmetical series.

EXERCISES

172. 1. Insert six arithmetical means between 7 and 63.

The six terms to be found, and the given numbers 7 and 63, will make 8 terms.

We have $n = 8$, $a = 7$, $l = 63$. Substitute in (A),

$$63 = 7 + (8-1)d,$$
$$63 = 7 + 7d,$$
$$d = 8.$$

Hence the required series is 7, 15, 23, 31, 39, 47, 55, 63.

2. Insert 5 arithmetical means between 6 and 72.

3. Insert 4 arithmetical means between 7 and -23.

4. Insert 8 arithmetical means between 7 and $8\frac{1}{2}$.

5. Insert 3 arithmetical means between x and y.

6. Insert 7 arithmetical means between $\sqrt{7}$ and $10\sqrt{7}$.

7. Insert an arithmetic mean between 100 and 133.

SUM OF AN ARITHMETICAL SERIES

173. The sum of the terms of an arithmetical series can always be found by writing down all the terms, and adding them. But, if the number of terms is great, this operation is quite laborious. We proceed to derive a formula by which the sum of a large number of terms may be computed with less labor.

Observing that, if the last term is l, the term immediately preceding may be written $l - d$, the term before this $l - 2d$, and so on, we may indicate the sum of the series thus,

$$S = a + (a + d) + (a + 2d) + \ldots + (l - 2d) + (l - d) + l. \qquad (1)$$

Reverse the order of the terms in the right side of (1),

$$S = l + (l - d) + (l - 2d) + \ldots + (a + 2d) + (a + d) + a. \qquad (2)$$

Adding (1) and (2),

$$2S = (a+l) + (a+l) + (a+l) + \ldots + (a+l) + (a+l) + (a+l). \qquad (3)$$

In (3) there are as many parentheses $(a + l)$ as there are terms in the series; hence,

$$2S = n(a + l),$$

$$S = \frac{n}{2}(a + l). \qquad (B)$$

EXERCISES

174. 1. Find the sum of 20 terms of the series 99, 103, 107, 111, $\cdots$.

Here $n = 20,\ a = 99,\ d = 4.$

Hence, $l = a + (n-1)d = 99 + 19 \times 4 = 175,$

and $S = \frac{n}{2}(a+l) = \frac{20}{2}(99+175) = 10 \times 274 = 2740.$

The required sum is 2740.

2. Find the sum of 25 terms of the series $-50, -45, -40, \cdots$.

3. Find the sum of 18 terms of the series 11, 8, 5, 2, $\cdots$.

4. Find the sum of the first 200 integers, 1, 2, 3, $\cdots$.

5. Find the sum of all the integers between -50 and 75, excluding -50 and 75.

6. Find the sum of 15 terms in $1\frac{1}{3}, 1\frac{2}{3}, 2, 2\frac{1}{3}, \cdots$.

7. In formula (B) substitute for l its value as given in formula (A) and derive a formula for S which does not contain l.

8. Find the sum of the first one hundred odd numbers.

9. Find the sum of the first one hundred even numbers.

10. How many of the integers 1, 2, 3, $\cdots$ must be added to yield the sum 55?

Use the formula $S = \frac{n}{2}(2a + (n-1)d)$, derived in Ex. 7. Do both answers satisfy the conditions of the problem?

11. How many terms of the series 5, 4, 3, $\cdots$ are necessary to yield a sum 9?

Do both answers satisfy the conditions of the problem?

12. The second term of an arithmetical series is 11, the fifth term is 20; find the 14th term.

13. The first week a store was opened the expenses exceeded the income by \$ 52.25. The second week the loss was

\$ 41.75. If the improvement in the trade continued at the same rate, how much profit was made in 24 weeks?

14. A man enters an office at a salary of \$ 1200, which is increased annually \$ 75. How much will the firm pay him during 18 years?

15. The three formulas found below give the salaries offered by three companies to men entering their employ. S is the monthly salary in dollars earned after a given number of years (n). Calculate which company after 20 years' employ will give the highest salary. How much does a man earn in 20 years in each case?

(*a*) $S = 100 + \frac{5}{2}n$, (*b*) $S = 95 + 4n$, (*c*) $S = 88 + \frac{13}{2}n$.

GEOMETRICAL SERIES

175. The series 3, 6, 12, 24, is not arithmetical; the difference between successive terms is not the same. The successive terms are formed in accordance with a different law. It is readily seen that any term after the first is derived from the preceding one by multiplying by 2. Such a series is called a *geometrical series* or a *geometrical progression.*

A geometric series is a succession of numbers in which each number after the first, when divided by the preceding number, always gives the same quotient.

The quotient is called the *common ratio.*

EXERCISES

176. Which of the following series are geometrical and which arithmetical?

1. $7, 35, 175, \cdots$.
2. $6, 12, 18, \cdots$.
3. $2, 8, 32, \cdots$.
4. $2, 8, 32, 40, \cdots$.
5. $2, -4, 8, -16, \cdots$.
6. $2, 1, 0, -1, \cdots$.
7. $\sqrt{2}, 1, \sqrt{\frac{1}{2}}, \cdots$.
8. $2^{\frac{1}{2}}, 2^{\frac{3}{2}}, 2^{\frac{5}{2}}, \cdots$.

LAST TERM OF A GEOMETRICAL SERIES

177. If the first term of a geometrical series is a, the common ratio is r, the number of terms n, then an expression for l, the last term, may be obtained by inspecting the terms in the general geometrical series,

$$a,\ ar,\ ar^2,\ ar^3,\ ar^4,\ \cdots.$$

We observe that in the second term, the exponent of r is 1, in the third term it is 2, in the fourth term it is 3, in the fifth term it is 4. Evidently, in the nth term, the exponent of r is $n-1$.

Hence we have the formula for the last term of a geometrical progression,

$$l = ar^{n-1}. \qquad (C)$$

EXERCISES

178. 1. Find the seventh term in the geometrical series, 16, 32, 64, $\cdots$.

Here $\quad a = 16,\ r = 2,\ n = 7.$

Hence, $\quad l = ar^{n-1} = 16(2)^{7-1} = 16 \times 64 = 1024.$

The seventh term is 1024. This result may be verified easily by writing down the first seven terms of the series.

2. Find the 8th term of the series 5, $2\frac{1}{2}$, $1\frac{1}{4}$, $\cdots$.
3. Find the 6th term of the series 130, 390, 1170, $\cdots$.
4. Find the 9th term of 3, -6, 12, -24, $\cdots$.
5. Find the 8th term of 3, -1, $\frac{1}{3}$, $-\frac{1}{9}$, $\cdots$.
6. Indicate the 15th term of the series $\dfrac{5}{\sqrt{2}}$, 1, $\dfrac{\sqrt{2}}{5}$, $\cdots$.

GEOMETRICAL MEANS

179. The geometrical means between two numbers are numbers which, together with the two given numbers as first and last terms, form a geometrical series.

If the two given numbers are 8 and $\frac{1}{2}$, then 4, 2, 1 are three geometrical means, because 8, 4, 2, 1, $\frac{1}{2}$ is a geometric series.

EXERCISES

180. 1. Insert four geometrical means between 5 and 160.

The two given numbers and the four means make together 6 terms. We have $a = 5$, $l = 160$, $n = 6$. We must find r.

By (C),
$$l = ar^{n-1},$$
$$160 = 5\,r^5.$$
$$r^5 = 32,$$
$$r = 2.$$

Hence the required geometrical series is 5, 10, 20, 40, 80, 160.

2. Insert six geometrical means between 10 and -1280.

Solve $r^7 = -128$ by trial.

3. Insert four geometrical means between $3\,a$ and $96\,a^6$.

4. Insert one geometrical mean between 133 and 1197. What is the geometrical mean between two numbers? What is the arithmetical mean between two numbers?

5. Two numbers differ by 6, and their arithmetical mean exceeds their geometric mean by 1. Find the numbers.

SUM OF A GEOMETRICAL SERIES

181. The sum of the first n terms of a geometrical series may be indicated thus,

$$S = a + ar + ar^2 + \cdots + ar^{n-3} + ar^{n-2} + ar^{n-1}. \qquad (1)$$

Multiply both sides of (1) by r,

$$rS = ar + ar^2 + ar^3 + \cdots + ar^{n-2} + ar^{n-1} + ar^n. \qquad (2)$$

Subtract (1) from (2) and observe that all the terms disappear in subtraction, except a and ar^n. We obtain,

$$rS - S = ar^n - a$$
$$S(r-1) = a(r^n - 1)$$
$$S = \frac{a(r^n - 1)}{r - 1}. \qquad (C)$$

EXERCISES

182. 1. Find the sum of six terms of the geometrical series $11, 22, 44, \cdots$.

Here $$a = 11,\ n = 6,\ r = 2.$$

Hence, $$S = \frac{a(r^n - 1)}{r - 1} = \frac{11(2^6 - 1)}{2 - 1} = \frac{11(64 - 1)}{1} = 11 \times 63 = 693.$$

2. Find the sum of seven terms of $1, \frac{1}{2}, \frac{1}{4}, \cdots$.

3. Find the sum of six terms of $1, -\sqrt{2}, 2, -2\sqrt{2}, \cdots$.

4. Find the sum of five terms of $x, -xy^2, xy^4, \cdots$.

5. Show that formula (C) may be written $S = \dfrac{a(1 - r^n)}{1 - r}$.

For what values of r is this form more convenient than (C)?

6. Show that (C) may be written $S = \dfrac{a - rl}{1 - r}$.

7. What will \$1000 amount to in four years, interest 3 %, compounded annually?

8. What will \$500 amount to in two years at 4 % annual interest, compounded semiannually?

9. What will \$700 amount to in eight years at 4 % annual interest, compounded semiannually?

We simplify this computation by the use of logarithms.

The amount is given by the expression $x = \$700\ (1.02)^{16}$.

$$\log 1.02 = 0.0086,$$
$$\log (1.02)^{16} = 16 \times 0.0086 = 0.1376$$
$$\log 700 = 2.8451$$
$$\log x = 2.9827$$
$$x = \$961.$$

This answer is only approximate. Had we used a six-place table of logarithms, instead of the four-place table, the answer would have come out \$960.95, correct to the nearest cent.

10. What sum of money, at 3 % interest, compounded annually, will amount to \$ 1000 in 20 years?

Let \$ x be the sum, in 20 years this amounts to \$ $x(1.03)^{20}$.

Hence the equation, $x(1.03)^{20} = 1000$

$$x = \frac{1000}{(1.03)^{20}}.$$

Log 1000 $= 3.0000$

Log 1.03 $= 0.0128$.

Log $(1.03)^{20} = 20 \times 0.0128$ $= 0.2560$

$\log x = \overline{2.7440}$,

$x =$ \$ 554.60.

The answer, correct to the nearest cent, is \$ 553.68.

11. What sum of money, at 4 % interest, compounded annually, will amount to \$ 695 in 10 years?

12. A sum of \$ 497 draws compound interest at $3\frac{1}{2}$ % for 7 years. Find its amount.

13. If \$ 1 could have drawn 5 % compound interest during the past 500 years, what would be its amount now?

14. Owing to the introduction of automobiles the number of horses in a town decreased in five years from 560 to 336. Assuming a constant annual rate of decrease, find the number after four more years.

15. Determine the present value of \$ 1000 due in 5 years, on the supposition that money can be invested so as to yield 3 % interest when compounded annually.

16. A man undertook to pay \$ 100 to a charity one year, \$ 90 the next year, $\frac{9}{10}$ of the latter sum the third year, and so on, until his death. He died after making 21 payments. What was the total of his gifts?

Use logarithms in the computation.

17. Another man pledged \$ 175 and promised to increase the amount by $\frac{1}{5}$ every year. What is the total of his donations after 15 years?

18. A man saved \$ 250 every year and invested it in a business that brought him 4 % per annum compound interest. He made these investments for six consecutive years. What was the total sum standing to his credit immediately after he paid in his last \$ 250?

19. A man's business is increasing at a uniform rate, and so rapidly that his income is doubled in two years. What is the rate of increase or *growth-factor?*

If at starting his business was \$ a, it was \$ ar at the end of the first year, and \$ ar^2 at the end of the second. We have then $ar^2 = 2a$, or $r = \sqrt{2}$. The growth-factor $= \sqrt{2}$.

20. What is the annual rate of increase, or annual growth-factor, if income increased uniformly and doubled in the course of four years?

21. In 4 years the population of a town increased from 2000 to 2662. Determine the growth-factor on the supposition that the rate of increase was uniform.

22. A young tree grew in 6 years from a height of 32 in. to a height of 20 ft. 3 in. What constant annual growth-factor would account for this increase?

23. The population of a mining town, now 10,000, is falling off at the rate of 1 % per annum. What will the population be 2 years from now?

24. Between the ages $9\frac{1}{2}$ and $14\frac{1}{2}$ the average height of boys increases from 50 to 60 inches. Find how tall a boy would be at $10\frac{1}{2}$, $11\frac{1}{2}$, $12\frac{1}{2}$, and $13\frac{1}{2}$ years of age, on the supposition that the growth-factor was the same for these years. Compare your results with the statistical averages which give the heights as 51.9, 53.6, 55.4, 57.5 inches, respectively.

Use logarithms.

INFINITE GEOMETRICAL SERIES

183. An important kind of series arises when the number of terms in it is no longer restricted to a fixed finite number, but is permitted to increase without end.

Consider the series $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$.

It has five terms. Imagine now that the number of terms is much larger, and that this number is steadily growing. We obtain a series which may be written,

$$1, \tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{8}, \tfrac{1}{16}, \tfrac{1}{32}, \tfrac{1}{64}, \ldots \text{ to infinity.} \qquad (1)$$

The number of terms is taken larger than any large finite number that we may name.

We say that the series is *infinite.* The characteristic property of it is that it has no last term. Whenever we try to seize upon any one term, say $\frac{1}{2^{100}}$, as being the last one, we can always go one step farther and write down the term $\frac{1}{2^{101}}$ which comes after it. Considerations of this sort compel us to accept the conclusion that there is no last term in such a series.

Another noticeable feature about the series (1) is that its terms decrease toward the right; that is, the terms farther from the starting point are always the smaller. The nth term may be written $\frac{1}{2^{n-1}}$.

By taking n sufficiently large, we may make this nth term smaller than any small number which we may name. For instance, the nth term will become smaller than $\frac{1}{100}$, if we take $n = 8$. In this case $\frac{1}{2^{n-1}}$ becomes $\frac{1}{2^7} = \frac{1}{128}$, which is less than $\frac{1}{100}$.

If you name some smaller number, say $\frac{1}{1000}$, then again a value of n can be chosen, so that the nth term becomes smaller than it. Take, $n = 11$, then $\frac{1}{2^{10}} = \frac{1}{2^7} \times \frac{1}{2^3} = \frac{1}{128} \times \frac{1}{8} = \frac{1}{1024}$,

which is less than $\frac{1}{1000}$. You may name a still smaller fraction, and again a value can be assigned to n in the expression for the nth term, which makes that nth term less than that small fraction. No matter how small a fraction is written down, we can always find some term in series (1) which is smaller than that fraction.

In such a case we say that the term $\frac{1}{2^{n-1}}$ *approaches zero as a limit, as n increases without end.*

SUM OF AN INFINITE GEOMETRICAL SERIES

184. We proceed now to find the sum of the infinite series $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^{n-1}}+\cdots$ to infinity. Examine formula (C); we see that

$$S=\frac{a(r^n-1)}{r-1}=\frac{a(1-r^n)}{1-r}=\frac{a-ar^n}{1-r}=\frac{a}{1-r}-\frac{ar^n}{1-r}.$$

We know that this series gives the sum correctly for any finite and fixed number of terms. What is the sum when the series becomes infinite?

In the infinite series $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots$ we have $r=\frac{1}{2}$, $a=1$. Other geometrical series may be written, but in all of them that we shall consider now, we shall assume that the *ratio r is numerically less than 1.* In that case the power r^n *becomes smaller for larger values of n and approaches the limit zero as n increases without end.*

That this is so, may become clearer by a second illustration. Let $r=\frac{2}{3}$, then $r^2=\frac{4}{9}$, $r^3=\frac{8}{27}$, $r^4=\frac{16}{81}$, $r^5=\frac{32}{243}$, and so on. The diminution in the value of the power is not as rapid now as in the case, $r=\frac{1}{2}$; nevertheless, a value of the integer n can always be found, such that r^n becomes less than any small fraction previously named. To make r^n less than $\frac{1}{100}$, we need only take $n=12$. Since

$$r^6=\frac{64}{729}, \text{ we get } r^{12}=r^6\cdot r^6=\frac{64\times 64}{729\times 729}=\frac{4096}{531441}.$$

In this fraction the numerator is seen to be less than the one-hundredth part of the denominator.

185. What happens to the fraction $\frac{ar^n}{1-r}$, when the integer n increases without end, and r is numerically less than 1? Bear in mind that a, the first term of the series, is a fixed number; r is also a fixed number, hence the denominator $1-r$ is a fixed number. Only the factor r^n changes, as n increases without end. And we have seen that, under these circumstances, r^n approaches the limit zero. Hence the numerator ar^n also approaches the limit zero. Hence, the entire fraction, $\frac{ar^n}{1-r}$, *approaches the limit zero as n increases without end.*

The series,

$$S = a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$$

which for a fixed, finite number of terms n has the sum

$$S = \frac{a}{1-r} - \frac{ar^n}{1-r},$$

has the sum, $$S = \frac{a}{1-r}, \qquad (D)$$

when the series becomes infinite, so that n increases without end, provided that r is numerically less than 1.

From what we have said it appears that by the "sum" of an infinite series we mean the value which the sum of the first n terms of the series *approaches as a limit*, when n increases without end.

When the *common ratio* r of the geometrical series is *not* numerically less than 1, the above reasoning does not apply. In that case r^n does not approach the limit 0 as n increases without end. In the series 1, 2, 4, 8, $\cdots$ $r = 2$, and $r^n = 2^n$; 2^n becomes greater and greater as n increases. Thus $2^2 = 4$, $2^3 = 8$, $2^4 = 16$, etc. It is seen that 2^n increases without limit

when n increases. Hence the fraction $\frac{ar^n}{1-r}$ does not approach the limit zero.

When in an infinite geometrical series the ratio r is numerically less than 1, so that the sum of the first n terms approaches or *converges* to $\frac{a}{1-r}$ as a limit, the series is said to be *convergent.*

Convergent series play a very important part, both in theoretical and in applied mathematics.

EXERCISES

186. 1. Find the sum of the infinite geometrical series,

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^{n-1}}+\cdots.$$

Here $a=1$, $r=\frac{1}{2}$.

Hence $S=\frac{a}{1-r}=\frac{1}{1-\frac{1}{2}}=2.$

The sum of the series is 2.

To convince himself of the correctness of this result the student may find the sum of 3, 4, 5, or more terms and see that the sum gets very close to 2, the closer when more terms are added.

Let S_n mean the sum of the first n terms, then we see that $S_2=1.5$, $S_3=1.75$, $S_4=1.875$, $S_5=1.9375$, $S_6=1.96875$. By *actual addition* we can approach to the limit 2 closer and closer, but we cannot reach it, because we can perform this addition for only a finite number of terms.

2. Find the sum of the infinite geometrical series $1, \frac{2}{3}, \frac{4}{9}, \frac{8}{27}, \cdots$.

3. Find the sum of the infinite geometrical series $1, \frac{1}{10}, \frac{1}{100}, \frac{1}{1000}, \cdots$.

4. Find the sum of the infinite geometrical series $2, -\frac{2}{3}, \frac{2}{9}, -\frac{2}{27}, \cdots$.

5. Find the sum of the infinite geometrical series $1, \frac{1}{2^{\frac{1}{2}}}, \frac{1}{2}, \frac{1}{2^{\frac{3}{2}}}, \frac{1}{2^2}, \cdots$.

6. Find a common fraction which expresses the exact value of the repeating decimal, $.252525\cdots$.

The repeating fraction may be written as an infinite geometrical series thus,

$$.2525\cdots = \tfrac{25}{100} + \tfrac{25}{10000} + \tfrac{25}{1000000} + \cdots.$$

Here $\quad a = \tfrac{25}{100},\ r = \tfrac{25}{10000} \div \tfrac{25}{100} = \tfrac{1}{100}.$

Hence $\quad S = \dfrac{a}{1-r} = \dfrac{\frac{25}{100}}{1-\frac{1}{100}} = \dfrac{25}{100} \div \dfrac{99}{100} = \dfrac{25}{99}$, *Ans.*

Find the common fraction which is the exact value of each of the following repeating decimals:

7. $.333\cdots$. 8. $.151515\cdots$. 9. $7.363636\cdots$.

10. $.555\cdots$. 11. $.325325\cdots$. 12. $7.8212121\cdots$.

13. A man undertook to pay \$500 to a charity one year, \$450 the next year, $\frac{9}{10}$ of the latter sum the third year, and so on, until his death. What was the outside limit of the expectation of this charity? If the man died after making 15 donations, how much did his total payments fall short of that sum?

Use logarithms in performing the computation.

THEORY OF LIMITS

187. In deriving the sum of an infinite converging geometrical series we touched upon the theory of limits which plays a fundamental rôle in higher mathematics. The ideas which we brought out suggest the following definition of the limit of a varying number.

The limit of a variable is a fixed number which the variable approaches in such a way that the difference between the variable and its limit **becomes and remains** *numerically less than any number, however small, but never becomes zero.*

Thus, in $\quad S_n = 1 + \dfrac{1}{2} + \dfrac{1}{4} + \cdots + \dfrac{1}{2^{n-1}},$

the variable is S_n, which increases when the integer n increases.

The limit of S_n is the number 2.

As n increases without end, the variable S_n approaches 2 and may be made to approach it so closely that $2 - S_n$ is less than any previously assigned small number different from zero, such as $\frac{1}{10000}$ or $\frac{1}{1000000}$.

A variable of some importance is the fraction $\frac{1}{n}$, when n takes in succession the values $1, 2, 3, 4, \cdots$. It is seen that the variable $\frac{1}{n}$ becomes smaller and smaller and approaches the limit zero. Under the same conditions the fraction $\frac{a}{n}$, where a is any fixed number, will likewise approach the limit 0 as the integer n increases without end.

Quite different is the behavior of $\frac{a}{x}$, when a is a fixed number different from zero, and x takes in succession the values $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots$. For simplicity, let $a = 1$.

Then, for $x = 1$, $\frac{1}{x} = 1$, for $x = \frac{1}{8}$, $\frac{1}{x} = 8$,

for $x = \frac{1}{2}$, $\frac{1}{x} = 2$, for $x = \frac{1}{16}$, $\frac{1}{x} = 16$,

for $x = \frac{1}{4}$, $\frac{1}{x} = 4$, for $x = \frac{1}{32}$, $\frac{1}{x} = 32$,

and so on. The smaller the denominator, the larger the value of the fraction.

We let the variable x approach the limit zero in such a way that it does not reach zero. We stipulate that x shall not become zero for the reason that *we cannot divide by zero*, as was explained in a previous chapter. If, then, x is permitted to approach 0 as a limit, without reaching its limit, then the fraction $\frac{1}{x}$ increases in value without limit. That is, we can select a value for x, such that $\frac{1}{x}$ will be greater than any previously assigned large number.

For instance, select the number 1000; we can take x so small that $\frac{1}{x}$ is greater than 1000. Let $x = \frac{1}{2^{10}} = \frac{1}{2^5 \cdot 2^5} = \frac{1}{32 \cdot 32} = \frac{1}{1024}$, then $\frac{1}{x} = 1024$, which is more than 1000. In the same way, x can be so chosen, that $\frac{1}{x}$ exceeds 10,000, or any other fixed number. Hence the theorem,

As the variable x is made to approach the limit zero, without reaching zero, $\frac{a}{x}$ increases so as to exceed any large number that may be previously assigned.

This theorem is sometimes condensed in this manner:

As x approaches zero, $\frac{a}{x}$ approaches infinity.

A still further compression of the theorem consists in the use of the symbolism

$$\frac{a}{0} = \infty .$$

This symbolism is objectionable, because it may convey the idea that division by 0 is permissible. The reader must remember that these symbols are intended merely as a convenient abbreviation for the theorem given above.

EXERCISES

188. 1. What value does the fraction $\frac{a+x}{b-x}$ approach, when a and b are fixed numbers ($b \neq 0$) and x is a variable approaching the limit zero?

Under these conditions the numerator $a + x$ approaches a, the denominator $b - x$ approaches b.

Hence the fraction approaches $\frac{a}{b}$.

2. What value does the fraction $\frac{a-x}{a+x}$ approach, when x increases without limit?

The reasoning is easier, if we first divide numerator and denominator by x, so that x occurs in the denominators of the minor fractions.

We obtain,
$$\frac{a-x}{a+x}=\frac{\frac{a}{x}-1}{\frac{a}{x}+1}.$$

Since a is a fixed number, and x increases without limit, the fraction $\frac{a}{x}$ approaches the limit zero, hence, the numerator $\frac{a}{x}-1$ approaches -1 as a limit; the denominator $\frac{a}{x}+1$ approaches $+1$ as a limit; the complex fraction approaches $\frac{-1}{+1}$ or -1 as a limit.

What is the limiting value of the following fractions when x approaches zero as a limit?

3. $\frac{2+3x}{4-5x}$.

4. $\frac{x+5}{x+6}$.

5. $\frac{5}{1-x}$.

6. $\frac{3+x^2}{5+x^3}$.

7. $\frac{x+x^2}{2-x^2}$.

8. $\frac{x^3+x^2+1}{x^4+2x^2+3}$.

What is the limiting value of the following fractions when the variable x increases without limit?

9. $\frac{5}{x}$.

10. $\frac{5}{x+6}$.

11. $\frac{1+x}{1+2x}$.

12. $\frac{a+bx}{c+dx}$.

13. $\frac{a+x^2}{a-x^2}$.

14. $\frac{cx^2+x+a}{x^2-x+b}$.

SUPPLEMENT

THE HIGHEST COMMON FACTOR BY THE METHOD OF DIVISION

189. The method of finding the h. c. f. of two polynomials not easily factored is similar to the method for finding the h. c. f. of two numbers in arithmetic, neither of which can be easily factored. The method was first suggested by Euclid (300 B.C.).

EXAMPLE

Find the h. c. f. of 1343 and 3002.

```
1343)3002|2
     2686
      316)1343|4
          1264
            79)316|4
               316
```

Process: Divide 3002 by 1343.
Take the remainder 316 as a new divisor, and 1343 as a new dividend.
Take the next remainder 79 as a new divisor, and 316 as a new dividend.
The next remainder is zero.
The last divisor 79 is the h. c. f. of 1343 and 3002.

Proof: Let $A = 1343$, $R = 316$.
$B = 3002$, $Q = 2$.

We have, as always,

$$\text{Dividend} = \text{Divisor} \times \text{Quotient} + \text{Remainder}.$$

$$B = A \times Q + R,$$

or

$$R = B - AQ$$

From the last equation we see that any factor of both A and B must be also a factor of $B - AQ$, and therefore a factor of the remainder R. Hence, *all common factors of A and B are common factors of A and R.*

Instead of finding the h. c. f. of A and B, we may therefore find the h. c. f. of A and R; the latter process involves smaller numbers.

Repeating the above process, divide A by R. Let Q' be the new quotient 4, and R' be the new remainder 79. As before, all common factors of A and R are factors of R'; for, $A = Q'R + R'$ and $R' = A - Q'R$.

Hence, the problem reduces itself further, to finding the h. c. f. of R and R'; that is, to finding the h. c. f. of 316 and 79. As $316 = 79 \times 4$, it follows that 79 is the h. c. f. required.

This process may be used in finding the h. c. f. of polynomials which cannot be easily factored. Numerical factors common to the polynomials are easily detected by inspection. Such numerical common factors, as well as other common factors that can be found by inspection, should be removed at once.

EXAMPLE

Find the h. c. f. of $2x^3-22x-12$ and $4x^3+16x^2+16x+12$.

Process:

$$2x^3 - 22x - 12 = 2(x^3 - 11x - 6).$$
$$4x^3 + 16x^2 + 16x + 12 = 4(x^3 + 4x^2 + 4x + 3).$$

The numerical factor 2 is common to the polynomials.

$$
\begin{array}{l}
x^3 - 11x - 6)\,x^3 + 4x^2 + 4x + 3 \,\underline{|\,1} \\
\qquad\qquad\quad x^3 \qquad\quad - 11x - 6 \\
\hline
\qquad\qquad\qquad 4x^2 + 15x + 9)\,4x^3 - 44x - 24 \quad \underline{|\,x - 15} \\
\qquad\qquad\qquad\qquad\qquad\qquad 4x^3 + 9x + 15x^2 \\
\hline
\qquad\qquad\qquad\qquad\qquad\qquad -15x^2 - 53x - 24 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 4 \\
\hline
\qquad\qquad\qquad\qquad\qquad\qquad -60x^2 - 212x - 96 \\
\qquad\qquad\qquad\qquad\qquad\qquad -60x^2 - 225x - 135 \\
\hline
\qquad\qquad\qquad\qquad\qquad\qquad\quad 13 \,\underline{|\,13x + 39} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad x + 3)\,4x^2 + 15x + 9 \,\underline{|\,4x + 3} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 4x^2 + 12x \\
\hline
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 3x + 9 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 3x + 9 \\
\hline
\end{array}
$$

The h. c. f. $= 2(x + 3)$.

To avoid fractions any expression may be multiplied or divided by any number which is not a factor of the other. In the example above, when $4x^2$ in the first remainder will not be contained in x^3 a whole number of times, the expression $x^3 - 11x - 6$ is multiplied by 4. Again when $4x^2$ is not contained in $-15x^2$ a whole number of times, the

expression $-15x^2-53x-24$ is multiplied by 4. In the remainder $13x+39$, the factor 13 may be discarded because the expression $4x^2+15x+9$ does not have a factor 13. It would be wrong to multiply $4x^2+15x+9$ by 13, because 13 is a factor of $13x+39$.

Continue to divide until the remainder is of a lower degree than the divisor. Then take the remainder for a new divisor and the previous divisor for a new dividend and proceed as before.

The l. c. m. of two expressions not easily factored may be found by first finding their h. c. f. The l. c. m. required will then be the product of either expression and the quotient obtained by dividing the other expression by the h. c. f. It is seldom that resort to this long process becomes necessary.

EXERCISES

190. Find h. c. f. of:

1. $2a^2+a-3$ and $4a^3+8a^2-a-6$.
2. $2x^3-5x^2-2x+2$ and $2x^3-7x^2+9x-3$.
3. $12y^4+30y^3-72y^2$ and $32y^3+84y^2-176y$.
4. $a^4-a^3+2a^2-a+1$ and $a^4+a^3+2a^2+a+1$.
5. $8am^3+24am^2+22am+6a$ and $6m^3+13m^2+8m+1$.
6. $2x^3-10x^2-14x+70$ and $x^3-3x^2-7x-15$.
7. $8x^4y-10x^3y+7x^2y-2xy$ and $6x^5y-11x^4y+8x^3y-2x^2y$.
8. $4a^3-4a^2-5a+3$ and $10a^2-19a+6$.

The h. c. f. of three or more expressions may be found as follows: Find the h. c. f. of two of them; then find the h. c. f. of this result and another of the given expressions, and so on. The last h. c. f. is the one required.

9. $2x^3-2x^2-2x-4$, $3x^3-6x^2+9x-18$, and $8x^3-12x^2-4x-8$.
10. $a^3x-6a^2x+11ax-6x$, $a^3x-9a^2x+26ax-24x$, and $a^3x^2-8a^2x^2+19ax^2-12x^2$.

SOLUTIONS OF QUADRATIC EQUATIONS

191. In previous solutions of the quadratic equation $ax^2 + bx + c = 0$, obtained by completing the square, the first step was to divide both sides by a. There are other methods of procedure which possess certain advantages. If a, b, c are integers and it is desired to avoid fractions in the process of completing the square, the *Hindu method* of completing the square may be used. It is as follows:

Given $$ax^2 + bx = -c.$$

Multiply both sides by *four times the coefficient of* x^2, or $4a$.

We obtain $$4a^2x^2 + 4abx = -4ac.$$

Divide $4abx$ by twice the square root of $4a^2x^2$; this gives b.

Add the square of b to both sides,

$$4a^2x^2 + 4abx + b^2 = b^2 - 4ac.$$

Take the square root of both sides,

$$2ax + b = \pm\sqrt{b^2 - 4ac}.$$

Transpose b and divide by $2a$, $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Sometimes fractions may be avoided by the use of a smaller factor than $4a$. Consider the equation,

$$3x^2 + 4x = 8.$$

Make the first term a perfect square by multiplying by 3,

$$9x^2 + 12x = 24.$$

Divide $12x$ by twice the square root of $9x^2$; this gives 2.

Add the square of 2 to both sides,

$$9x^2 + 12x + 4 = 28.$$

Then $$3x + 2 = \pm\sqrt{28},$$

$$x = \frac{-2 \pm \sqrt{28}}{3},$$

$$x = \frac{-2 \pm 2\sqrt{7}}{3}.$$

Solve by completing the square without introducing fractions:

1. $2x^2 - 13x + 10 = 0.$
2. $5x^2 + 7x - 13 = 0.$
3. $4x^2 + 6x - 41 = 0.$
4. $7x^2 - 14x - 3 = 0.$

MATHEMATICAL INDUCTION AND PROOF OF THE BINOMIAL THEOREM

192. There is an important method of reasoning in mathematics, called *mathematical induction*, which we shall use in proving the binomial formula. The method will be grasped more readily, if we give a non-mathematical illustration of it.

Non-mathematical Illustration.

For the sake of argument, suppose it established as true

(1) *That there was once an Indian named Hiawatha.*

(2) *That every Indian named Hiawatha (if ever such an Indian existed) had a grown son named Hiawatha.*

From these two propositions certain conclusions can be drawn.

By (1) we know that an Indian named Hiawatha once existed.

By (2) we know that he had a grown son named Hiawatha. Designate him Hiawatha II.

By (2) we know that Hiawatha II had himself a grown son named Hiawatha. Designate him Hiawatha III.

By (2) we know that Hiawatha III had himself a grown son named Hiawatha. And so on.

Thus the inference is drawn from propositions (1) and (2) that there existed an unbroken, never-ending line of descent of Indians named Hiawatha.

Mathematical Illustration

It is to be proved that *the sum of the first n positive integers is* $\frac{n}{2}(n+1)$.

I. We see *by trial* that

$$1+2=\tfrac{2}{2}(2+1),$$
$$1+2+3=\tfrac{3}{2}(3+1).$$

This shows that the theorem is correct for the particular cases $n=2$ and $n=3$.

II. We establish the conditional proposition, that *if* the theorem is true for some value of n, say $n=m$, then the theorem must be true for $n=m+1$.

If the theorem is true for $n=m$, we have

$$1+2+3+\cdots+m=\frac{m}{2}(m+1).$$

Add $(m+1)$ to both sides,

$$1+2+3+\cdots+m+(m+1)=\frac{m}{2}(m+1)+(m+1).$$

Combine terms on the right side,

$$1+2+3+\cdots+m+(m+1)=\frac{m+1}{2}([m+1]+1).$$

As this result is the original theorem for $n=m+1$, we have shown that the theorem is true for $n=m+1$, *provided* it is true for $n=m$.

Next we apply the reasoning called *mathematical induction:*

By I, the formula

$$1+2+3+\cdots+n=\frac{n}{2}(n+1)$$

is true for $n=3$.

By II it follows that, being true for $n=3$, it must be true also for $n=4$.

By II it follows that, being true for $n=4$, it must be true also for $n=5$. And so on for $n=6, 7, 8, \cdots$. Consequently, the theorem is true for any positive integral value of n, no matter how great it may be. Thus, the theorem is established.

NOTE. Very often a beginner fails to perceive the need of such an involved argument. If a theorem is true for certain special cases, he may conclude at once, without further argument, that it is true for all cases. But such jumping at conclusions is dangerous. The relation $a^2=2a$ holds for $a=0$, also for $a=2$; is it true generally? Is it true for any other value of a?

PROOF OF THE BINOMIAL THEOREM WHEN THE EXPONENT OF THE BINOMIAL IS A POSITIVE INTEGER

193. By actual multiplication it is found that

$$(a\pm b)^2=a^2\pm 2ab+b^2,$$

$$(a\pm b)^3=a^3\pm 3a^2b+3ab^2\pm b^3,$$

$$(a\pm b)^4=a^4\pm 4a^3b+6a^2b^2\pm 4ab^3+b^4.$$

By inspection we find that these products follow the following laws:

I. The first term is a raised to the same power as that of the binomial. In each succeeding term the exponent of a decreases by 1.

II. The factor b does not appear in the first term; the exponent of b in the second term is 1 and increases by 1 in each succeeding term.

III. The coefficient of any term after the first is found by multiplying the coefficient of the preceding term by the exponent of a in that term, and divided by one more than the exponent of b.

IV. If the binomial is $a + b$, the signs of the product are all plus; if the binomial is $a - b$, the signs are alternately $+$ and $-$.

V. The number of terms is one more than the exponent of the binomial.

VI. Each term is of the same degree as the binomial.

If we use the exponent n, these six laws are embodied in the following formula: *

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \cdots + nab^{n-1} + b^n. \quad (1)$$

Thus far, this formula has been shown to be true only for the particular cases $n = 2$, 3, or 4. (Substitute 4 for n and show that the product of $(a + b)^4$ is obtained.) Is formula (1) true for all positive integral values of n? Were we to attempt to prove its generality by actual multiplication, we would soon weary; the task would be impossible. By mathematical induction the proof is short.

Let us establish the conditional proposition that *if* formula (1) holds for, say $n = m$, it must hold also for $m + 1$. In (1) write m in place of n. Then multiply both sides of the resulting equation by $(a + b)$. We obtain,

* By the notation $2!$ we mean 2×1, by $3!$ we mean $3 \times 2 \times 1$; generally, $n! = n(n-1)(n-2) \ldots 3.2.1$. We call $n!$ "factorial n."

$$(a+b)^m = a^m + ma^{m-1}b + \frac{m(m-1)}{2!}a^{m-2}b^2 + \cdots + mab^{m-1} + b^m, \quad (2)$$

$$\begin{array}{l} (a+b) = a + b \\ \hline (a+b)^{m+1} \\ = \left\{ \begin{array}{l} a^{m+1} + ma^m b + \frac{m(m-1)}{2!}a^{m-1}b^2 + \cdots + \qquad ma^2b^{m-1} + \quad ab^m \\ \qquad a^m b + \qquad ma^{m-1}b^2 + \cdots + \frac{m(m-1)}{2!}a^2b^{m-1} + mab^m + b^{m+1} \end{array} \right. \\ \hline \end{array}$$

$$(a+b)^{m+1} = a^{m+1} + (m+1)a^m b + \frac{(m+1)m}{2!}a^{m-1}b^2 + \cdots + \frac{(m+1)m}{2!}a^2b^{m-1}$$
$$+ (m+1)ab^m + b^{m+1}. \quad (3)$$

The operation given above should be studied very carefully. Several steps require explanation. We have multiplied both sides of equation (2) by $a + b$. Examine every step carefully. In the right sides we first multiplied by a, then by b. The dots stand for terms that could not all be written down, because the number of terms is $m + 1$, and m is not given in Hindu-Arabic numerals. The term $\frac{m(m-1)}{2!}a^{m-2}b^3$ which is obtained when we multiply by b, is not written down and is among the terms represented by dots. The term $\frac{m(m-1)}{2!}a^2b^{m-1}$ is obtained by multiplying by b the term $\frac{m(m-1)}{2!}a^2b^{m-2}$ which in (2) lies among the terms represented by dots.

Be sure to verify the addition of the partial products.

We observe that (3) is the same in form as (1); that is, if in (1) we write $m + 1$ for n, we obtain (3).

By the above multiplication we have proved that the formula (1) is true for $n = m + 1$, *provided* it is true for $n = m$.

Now we proceed to the argument by mathematical induction:

Formula (1) is known from actual multiplication to be true for $n = 4$. But we have shown that if (1) is true for $n = 4$, it must be true for $n = 5$.

Again, if (1) is true for $n = 5$, it must be true for $n = 6$. And so on, for $n = 6, 7, 8, \cdots$.

Hence formula (1) is true for any positive integral value of the exponent.

EXERCISES

194. 1. Expand $(x+3y)^6$.

In formula (1) substitute $a = x$, $b = (3y)$, $n = 6$. We obtain,

$$(x+3y)^6 = x^6 + 6x^5(3y) + \frac{6\cdot 5}{2}x^4(3y)^2 + \frac{6\cdot 5\cdot 4}{3\cdot 2}x^3(3y)^3$$
$$+ \frac{6\cdot 5\cdot 4\cdot 3}{4\cdot 3\cdot 2}x^2(3y)^4 + \frac{6\cdot 5\cdot 4\cdot 3\cdot 2}{5\cdot 4\cdot 3\cdot 2}x(3y)^5 + (3y)^6.$$

Simplifying,

$$(x+3y)^6 = x^6 + 18x^5y + 135x^4y^2 + 540x^3y^3 + 1215x^2y^4 + 1458xy^5 + 729y^6.$$

2. Expand $(2p-q)^5$.

In formula (1) substitute $a = 2p$, $b = (-q)$, $n = 5$.

$$(2p-q)^5 = (2p)^5 + 5(2p)^4(-q) + \frac{5\cdot 4}{2}(2p)^3(-q)^2$$
$$+ \frac{5\cdot 4\cdot 3}{3\cdot 2}(2p)^2(-q)^3 + \frac{5\cdot 4\cdot 3\cdot 2}{4\cdot 3\cdot 2}(2p)(-q)^4 + (-q)^5.$$

Simplifying,

$$(2p-q)^5 = 32p^5 - 80p^4q + 80p^3q^2 - 40p^2q^3 + 10pq^4 - q^5.$$

Expand:

3. $(a-2b)^5$.
4. $(x-\frac{1}{2}y)^4$.
5. $(3x-\frac{1}{2}z)^4$.
6. $\left(\frac{1}{m}-n\right)^6$.
7. $(2p+3q)^5$.
8. $(\sqrt{2}\,a-b)^4$.
9. $(\sqrt{2}\,x+\sqrt[3]{2}\,y)^3$.
10. $(a-b)^7$.
11. $(a+b)^8$.
12. $(a+\sqrt{-1}\,b)^3$.
13. $(a-\sqrt{-1}\,b)^4$.
14. $(\sqrt{-1}\,x+\sqrt{-1}\,y)^4$.

195. It can be proved that, under certain limitations, the Binomial Formula is applicable to cases in which the exponent n of $(a+b)^n$ is not a positive integer, but is a negative integer, or a positive or negative fraction. In such cases the binomial expansion becomes an *infinite series.* Such an infinite series can be used for purposes of computation only when it is *convergent;* that is, only when the sum of the first r terms of the series approaches a finite constant as a limit, as r increases without end. When $a > b$, the expansion, expressed in ascending powers of b, is always a convergent series. Omitting all proofs, we proceed to applications.

1. Find approximately the cube root of $27\frac{1}{5}$.

This can be obtained by the regular process of extracting the cube root, or by the use of logarithms. A third method is by means of the binomial theorem.

We notice that $(27.2)^{\frac{1}{3}}$ is a little over 3. Take $27 \cdot 2 = 27 + .2$, where 27 is a perfect cube. Then by the binomial formula, taking

$$a = 27,\ b = .2,\ n = \tfrac{1}{3},$$

$$(27 + .2)^{\frac{1}{3}} = (27)^{\frac{1}{3}} + \tfrac{1}{3} \cdot (27)^{-\frac{2}{3}}(.2) + \frac{\frac{1}{3}(\frac{1}{3} - 1)}{2}(27)^{-\frac{5}{3}}(.2)^2 + \ldots$$

Here .2 is small when compared with 27. Hence the series is convergent.

In the expansion, .2 occurs to the first power in the second term, to the second power in the third term, and to higher powers in the terms following which are not put down.

The simplified value of the second term is $+ \frac{1}{135} = .007407$, of the third term is $- \frac{1}{54675} = - .000018$.

If we take only the first two terms in the expansion, and neglect all the rest, we obtain an approximate value for the cube root, 3.007407.

If we take the first three terms, the more accurate value, 3.007389, is obtained.

If in a calculation, account is taken of a number b which is small compared with the other numbers involved, but the square and higher powers of b are discarded, then the calculation is said to be carried to the *first approximation*.

If both the first and second powers of b are used, but no higher powers of b, then the calculation is carried to the *second approximation*. And so on.

In the above example, where $b = .2$, the *first approximation* to the cube root of $27\frac{1}{5}$ is 3.007407 ; the *second approximation* is 3.007389.

EXERCISES

196. Find to a first approximation the values of the following expressions in which the second number is much smaller than the first:

1. $(32\frac{1}{4})^{\frac{1}{5}}$.
2. $(63)^{\frac{1}{6}} = (64 - 1)^{\frac{1}{6}}$.
3. $\sqrt{99} = \sqrt{100 - 1}$.
4. $(a^2 + b)^{\frac{1}{2}}$.
5. $\sqrt{a^2 + 2c}$.
6. $\sqrt[3]{4p + q}$.
7. $(a^2 - 3b)^{\frac{1}{5}}$.
8. $(9m^2 - n)^{\frac{1}{7}}$.
9. $\sqrt[3]{a^3 - \sqrt[4]{a}}$.

10. A brass cube 1 inch each way is heated until its edges are 1.003 inches long. Compute to the first approximation the area of each face and the volume of the cube.

11. A metal cube 2 inches each way is cooled so that its volume is reduced to 7.998 cubic inches. What is now the length of an edge?

12. Compute to a second approximation the amount after 10 years, of \$ 575 at 2 %, interest compounded annually.

Evaluate $575\,(1 + .02)^{10}$ by retaining the first and second powers of .02 in the binomial expansion, but discarding higher powers. Does the *first* approximation give results differing from those due to simple interest?

13. In measuring the diameter of a sphere whose diameter is actually 5″, an error of 1 % is probable. What is the probable error, computed to a first approximation, of the volume derived from the inaccurate diameter?

The volume of a sphere $= \frac{4}{3}\pi r^3$.
Take $r^3 = \frac{1}{8}(5 + .05)^3$.

14. How great an error is made by assuming

$$\frac{1 - a^7}{1 + a} = 1 - a + a^2 - a^3 + a^4,$$

when $a = \frac{1}{3}$? What is the ratio of the error to the whole of the fraction?

15. Proceed on the assumption that the earth is a sphere, the radius of which is 4000 miles. On the supposition that its radius was at one time 10 miles greater than at present, calculate to the first approximation, and also to the second approximation, the amount that it has lost during the contraction, (1) in superficial area, (2) in volume.

[The superficial area $= 4\,\pi r^2$.]

197. *A Table of Squares and Cubes, Square Roots and Cube Roots of Numbers from 1 to 200.*

Squares	Cubes	No.	Square Roots	Cube Roots	Squares	Cubes	No.	Square Roots	Cube Roots
1	1	1	1.000	1.000	2,601	132,651	51	7.141	3.708
4	8	2	1.414	1.260	2,704	140,608	52	7.211	3.733
9	27	3	1.732	1.442	2,809	148,877	53	7.280	3.756
16	64	4	2.000	1.587	2,916	157,464	54	7.348	3.780
25	125	5	2.236	1.710	3,025	166,375	55	7.416	3.803
36	216	6	2.449	1.817	3,136	175,616	56	7.483	3.826
49	343	7	2.646	1.913	3,249	185,193	57	7.550	3.849
64	512	8	2.828	2.000	3,364	195,112	58	7.616	3.871
81	729	9	3.000	2.080	3,481	205,379	59	7.681	3.893
100	1,000	10	3.162	2.154	3,600	216,000	60	7.746	3.915
121	1,331	11	3.317	2.224	3,721	226,981	61	7.810	3.936
144	1,728	12	3.464	2.289	3,844	238,328	62	7.874	3.958
169	2,197	13	3.606	3.351	3,969	250,047	63	7.937	3.979
196	2,744	14	3.742	2.410	4,096	262,144	64	8.000	4.000
225	3,375	15	3.873	2.466	4,225	274,625	65	8.062	4.021
256	4,096	16	4.000	2.520	4,356	287,496	66	8.124	4.041
289	4,913	17	4.123	2.571	4,489	300,763	67	8.185	4.061
324	5,832	18	4.243	2.621	4,624	314,432	68	8.246	4.082
361	6,859	19	4.359	2.668	4,761	328,509	69	8.307	4.102
400	8,000	20	4.472	2.714	4,900	343,000	70	8.367	4.121
441	9,261	21	4.583	2.759	5,041	357,911	71	8.426	4.141
484	10,648	22	4.690	2.802	5,184	373,248	72	8.485	4.160
529	12,167	23	4.796	2.844	5,329	389,017	73	8.544	4.179
576	13,824	24	4.899	2.884	5,476	405,224	74	8.602	4.198
625	15,625	25	5.000	2.924	5,625	421,875	75	8.660	4.217
676	17,576	26	5.099	2.962	5,776	438,976	76	8.718	4.236
729	19,683	27	5.196	3.000	5,929	456,533	77	8.775	4.254
784	21,952	28	5.292	3.037	6.084	474,552	78	8.832	4.273
841	24,389	29	5.385	3.072	6,241	493,039	79	8.888	4.291
900	27,000	30	5.477	3.107	6,400	512,000	80	8.944	4.309
961	29,791	31	5.568	3.141	6,561	531,441	81	9.000	4.327
1,024	32,768	32	5.657	3.175	6,724	551,368	82	9.055	4.344
1,089	35,937	33	5.745	3.208	6,889	571,787	83	9.110	4.362
1,156	39,304	34	5.831	3.240	7,056	592,704	84	9.165	4.380
1,225	42,875	35	5.916	3.271	7,225	614,125	85	9.219	4.397
1,296	46,656	36	6.000	3.302	7,396	636,056	86	9.274	4.414
1,369	50,653	37	6.083	3.332	7,569	658,503	87	9.327	4.431
1,444	54,872	38	6.164	3.362	7,744	681,472	88	9.381	4.448
1,521	59,319	39	6.245	3.391	7,921	704,969	89	9.434	4.465
1,600	64,000	40	6.325	3.420	8,100	729,000	90	9.487	4.481
1,681	68,921	41	6.403	3.448	8,281	753,571	91	9.539	4.498
1,764	74,088	42	6.481	3.476	8,464	778,688	92	9.591	4.514
1,849	79,507	43	6.557	3.503	8,649	804,357	93	8.644	4.531
1,936	85,184	44	6.633	3.530	8,836	830,584	94	9.695	4.547
2,025	91,125	45	6.708	3.557	9,025	857,375	95	9.747	4.563
2,116	97,336	46	6.782	3.583	9,216	884,736	96	9.798	4.579
2,209	103,823	47	6.856	3.609	9,409	912,673	97	9.849	4.595
2,304	110,592	48	6.928	3.634	9,604	941,192	98	9.900	4.610
2,401	117,649	49	7.000	3.659	9,801	970,299	99	9.950	4.626
2,500	125,000	50	7.071	3.684	10,000	1,000,000	100	10.000	4.642

Squares	Cubes	No.	Square Roots	Cube Roots	Squares	Cubes	No.	Square Roots	Cube Roots
10,201	1,030,301	**101**	10.050	4.657	22,801	3,442,951	**151**	12.288	5.325
10,404	1,061,208	**102**	10.100	4.672	23,104	3,511,808	**152**	12.329	5.337
10,609	1,092,727	**103**	10.149	4.688	23,409	3,581,577	**153**	12.369	5.348
10,816	1,124,864	**104**	10.198	4.703	23,716	3,652,264	**154**	12.410	5.360
11,025	1,157,625	**105**	10.247	4.718	24,025	3,723,875	**155**	12.500	5.372
11,236	1,191,016	**106**	10.296	4.733	24,336	3,796,416	**156**	12.490	5.383
11,449	1,225,043	**107**	10.344	4.747	24,649	3,869,893	**157**	12.530	5.395
11,664	1,259,712	**108**	10.392	4.762	24,964	3,944,312	**158**	12.570	5.406
11,881	1,295,029	**109**	10.440	4.777	25,281	4,019,679	**159**	12.610	5.418
12,100	1,331,000	**110**	10.488	4.791	25,600	4,096,000	**160**	12.649	5.429
12,321	1,367,631	**111**	10.536	4.806	25,921	4,173,281	**161**	12.689	5.440
12,544	1,404,928	**112**	10.583	4.820	26,244	4,251,528	**162**	12.728	5.451
12,769	1,442,897	**113**	10.630	4.835	26,569	4,330,747	**163**	12.767	5.463
12,996	1,481,544	**114**	10.677	4.849	26,896	4,410,944	**164**	12.806	5.474
13,225	1,520,875	**115**	10.724	4.863	27,225	4,492,125	**165**	12.845	5.485
13,456	1,560,896	**116**	10.770	4.877	27,556	4,574,296	**166**	12.884	5.496
13,689	1,601,613	**117**	10.817	4.891	27,889	4,657,463	**167**	12.923	5.507
13,924	1,643,032	**118**	10.863	4.905	28,224	4,741,632	**168**	12.961	5.518
14,161	1,685,159	**119**	10.909	4.919	28,561	4,826,809	**169**	13.000	5.529
14,400	1,728,000	**120**	10.954	4.932	28,900	4,913,000	**170**	13.038	5.540
14,641	1,771,561	**121**	11.000	4.946	29,241	5,000,211	**171**	13.077	5.550
14,884	1,815,848	**122**	11.045	4.960	29,584	5,088,448	**172**	13.115	5.561
15,129	1,860,867	**123**	11.091	4.973	29,929	5,177,717	**173**	13.153	5.572
15,376	1,906,624	**124**	11.136	4.987	30,276	5,268,024	**174**	13.191	5.583
15,625	1,953,125	**125**	11.180	5.000	30,625	5,359,375	**175**	13.229	5.593
15,876	2,000,376	**126**	11.225	5.013	30,976	5,451,776	**176**	13.266	5.604
16,129	2,048,383	**127**	11.269	5.027	31,329	5,545,233	**177**	13.304	5.615
16,384	2,097,152	**128**	11.314	5.040	31,684	5,639,752	**178**	13.342	5.625
16,641	2,146,689	**129**	11.358	5.053	32,041	5,735,339	**179**	13.379	5.636
16,900	2,197,000	**130**	11.402	5.066	32,400	5,832,000	**180**	13.416	5.646
17,161	2,248,091	**131**	11.446	5.079	32,761	5,929,741	**181**	13.454	5.657
17,424	2,299,968	**132**	11.489	5.092	33,124	6,028,568	**182**	13.491	5.667
17,689	2,352,637	**133**	11.533	5.104	33,489	6,128,487	**183**	13.528	5.677
17,956	2,406,104	**134**	11.576	5.117	33,856	6,229,504	**184**	13.565	5.688
18,225	2,460,375	**135**	11.619	5.130	34,225	6,331,625	**185**	13.601	5.698
18,496	2,515,456	**136**	11.662	5.143	34,596	6,434,856	**186**	13.638	5.708
18,769	2,571,353	**137**	11.705	5.155	34,969	6,539,203	**187**	13.675	5.718
19,044	2,628,072	**138**	11.747	5.168	35,344	6,644,672	**188**	13.711	5.729
19,321	2,685,619	**139**	11.790	5.180	35,721	6,751,269	**189**	13.748	5.739
19,600	2,744,000	**140**	11.832	5.192	36,100	6,859,000	**190**	13.784	5.749
19,881	2,803,221	**141**	11.874	5.205	36,481	6,967,871	**191**	13.820	5.759
20,164	2,863,288	**142**	11.916	5.217	36,864	7,077,888	**192**	13.856	5.769
20,449	2,924,207	**143**	11.958	5.229	37,249	7,189,057	**193**	13.892	5.779
20,736	2,985,984	**144**	12.000	5.241	37,636	7,301,384	**194**	13.928	5.789
21,025	3,048,625	**145**	12.042	5.254	38,025	7,414,875	**195**	13.964	5.799
21,316	3,112,136	**146**	12.083	5.266	38,416	7,529,536	**196**	14.000	5.809
21,609	3,176,523	**147**	12.124	5.278	38,809	7,645,373	**197**	14.036	5.819
21,904	3,241,792	**148**	12.166	5.290	39,204	7,762,392	**198**	14.071	5.828
22,201	3,307,949	**149**	12.207	5.301	39,601	7,880,599	**199**	14.107	5.838
22,500	3,375,000	**150**	12.247	5.313	40,000	8,000,000	**200**	14.142	5.848

REVIEW EXERCISES

(Selected from College Entrance Examinations)

FACTORING

198. 1. Factor $6a^2 + 10ab - 4b^2$, $x^3 + 125$, $1 - x - x^2 + x^3$, $x^8 + x^4 + 1$. *Univ. of State of N. Y.*

2. Find the prime factors of

(*a*) $(x - x^2)^3 + (x - 1)^3 + (1 - x)^3$.

(*b*) $(2x + a - b)^4 - (x - a + b)^4$. *Mass. Institute of Tech.*

3. Factor $x^2 - 2mx + m^2 - n^2$,

$x^{2n+1} - 11x^{n+1}y^m + 18xy^{2m}$. *Univ. of Pa.*

4. (*a*) Resolve the following into their prime factors:

(1) $(x^2 - y^2)^2 - y^4$.

(2) $10x^2 - 7x - 6$.

(*b*) Find the h. c. f. and the l. c. m. of

$x^3 - 3x^2 + x - 3$,

$x^3 - 3x^2 - x + 3$. *Columbia.*

5. Find the h. c. f.: $x^4 - y^4$,

$x^3 - xy^2 + x^2y - y^3$,

$x^4 + 2x^2y^2 - 3y^4$. *Cornell.*

6. Factor the following expressions:

(*a*) $a^{\frac{3}{4}} - b^{\frac{3}{4}}$,

(*b*) $x^2y^2z^2 - x^2z - y^2z + 1$,

(*c*) $16(x + y)^4 - (2x - y)^4$. *Mount Holyoke.*

FRACTIONS

199. 1. Simplify $\dfrac{a}{b^2} - \dfrac{a}{b^2 + \dfrac{cb}{a - \dfrac{c}{b}}}$. *Harvard.*

2. Simplify the expression

$$\left\{x + y - \frac{1}{x + y - \dfrac{xy}{x + y}}\right\} \frac{x^3 - y^3}{x^2 - y^2}.$$ *Cornell.*

3. Divide $\left(\dfrac{x^4 - y^4}{x^2 - y^2} \div \dfrac{x + y}{x^2 - xy}\right) \div \left(\dfrac{x^2 + y^2}{x - y} \div \dfrac{x + y}{xy - y^2}\right)$. *Sheffield.*

4. Simplify the expression

$$\left(1 - \frac{b^2}{a^2}\right)\left(1 - \frac{ab - b^2}{a^2}\right)\frac{a^4}{a^3 + b^3} \cdot \frac{a - b}{a^2 + b^2}.$$ *Mass. Inst. Tech.*

5. Find the product of

$$\left(1 + \frac{2a}{3} - \frac{5a^2}{6}\right) \text{ and } \left(2 - \frac{3a}{4} + \frac{a^2}{3}\right).$$ *Vassar.*

6. If $m = \dfrac{1}{a + 1}$, $n = \dfrac{2}{a + 2}$, $p = \dfrac{3}{a + 3}$, what is the value of $\dfrac{m}{1 - m} + \dfrac{n}{1 - n} + \dfrac{p}{1 - p}$? *Univ. of Pa.*

EQUATIONS

200. 1. Solve $\dfrac{x + a}{x + b} + \dfrac{x + b}{x + a} = \dfrac{5}{2}$. *Yale.*

2. Solve for x: $\dfrac{a - x}{\sqrt{a - x}} + \dfrac{x - b}{\sqrt{x - b}} = \sqrt{a - b}$. *Univ. of Pa.*

3. Solve for x: $\dfrac{2x+3}{x-1} - 6 = \dfrac{5}{x^2+2x-3}$. *Cornell.*

4. (*a*) Solve for x: $\sqrt{2x-3a} + \sqrt{3x-2a} = 3\sqrt{a}$.

(*b*) Solve for m:

$$1 - \frac{1}{2-m} = \frac{1}{m+2} + \frac{m-6}{4-m^2}.$$ *Univ. of Pa.*

5. Solve $\dfrac{1}{x+1} - \dfrac{1}{x-1} - \dfrac{1}{x-3} + \dfrac{1}{x-5} = 0$. *Princeton.*

6. Solve the equation $\sqrt{2x+5} - \sqrt{6-x} = 1$. *Univ. of Pa.*

7. Solve $3x^2 - 11x = 70$. *Univ. of State of N. Y.*

8. Solve the quadratic equation

$$x^2 - 1.6x + 0.3 = 0.$$ *Harvard.*

9. If d is one of the roots of the equation $ax^2 + bx + c = 0$, find the other root. *Univ. of Pa.*

10. Solve $\dfrac{6}{x+1} + \dfrac{2}{x} = 3$. *Univ. of State of N. Y.*

11. Solve $\sqrt{x+a} + \sqrt{x} + \sqrt{x-a} = 0$. *Univ. of State of N. Y.*

12. (i) Solve for t: $(t-2)(t+3) = (3t+4)(2-t)$.

(ii) Solve for x:

$$\frac{1}{\sqrt{x+2}} + \frac{1}{\sqrt{3x-2}} = \frac{4}{\sqrt{3x^2+4x-4}}.$$ *Univ. of Pa.*

RADICALS

201. 1. Find the square root of

$$x^6 - 2x^5 - x^4 + 3x^2 + 2x + 1. \quad \textit{Univ. of State of N. Y.}$$

2. Simplify $\left(\frac{x+1}{x-1}\sqrt{\frac{x-1}{x+1}}\right)(\sqrt{x^2-1})$, $\frac{(\sqrt[5]{a}\sqrt[4]{b^2})^3}{\sqrt[3]{c^2}} \div \frac{\sqrt[6]{c^2b^4}}{\sqrt[5]{a^7}}$.

Univ. of State of N. Y.

3. Solve for x and y the equations

$$\sqrt{x+y} = a + b,$$
$$x - y = (a - b)\sqrt{x+y}. \quad \textit{Univ. of Pa.}$$

4. Simplify $(x+1)^{\frac{3}{2}} - x\sqrt{x+1}$, $\sqrt{-m^3n} \cdot \sqrt{-mn^3}$,

$$\sqrt{\left[\frac{a^{\frac{1}{m}}}{b^{\frac{1}{n}}}\right]^{2mn}}, \left(\frac{3\sqrt[4]{x^3}}{5\sqrt{a^{-3}}}\right)^2. \quad \textit{Univ. of Pa.}$$

5. Simplify the product of

$$(ayx^{-1})^{\frac{1}{2}},\ (bxy^{-2})^{\frac{1}{3}}, \text{ and } (y^2a^{-2}b^{-2})^{\frac{1}{4}}. \quad \textit{Princeton.}$$

6. Solve the simultaneous equations

$$\begin{cases} x^{-\frac{1}{2}} + 2y^{-\frac{1}{2}} = \frac{7}{6}, \\ 2x^{-\frac{1}{2}} - y^{-\frac{1}{2}} = \frac{2}{3}. \end{cases} \quad \textit{Yale.}$$

7. Simplify (*a*) $\sqrt{6 - \sqrt{20}}$. (*b*) $\frac{1 + \sqrt{x^2+1}}{1 + \sqrt{x^2+1} + x^2}$.

Cornell.

8. Find the value of a^7b^7, if

$$a = x^{\frac{3}{7}}y^{-\frac{3}{2}} \text{ and } b = \tfrac{1}{2}x^{-\frac{1}{7}}y^{\frac{1}{2}},$$

and reduce the result to a form having only positive exponents.

Harvard.

9. Simplify $\frac{\sqrt{2} + 2\sqrt{3}}{\sqrt{2} - \sqrt{12}}$, and compute the value of the fraction to two decimal places. *Yale.*

SYSTEMS OF LINEAR EQUATIONS

202. 1. Solve $\frac{1}{x}-\frac{1}{y}=a, \frac{1}{y}+\frac{1}{z}=b, \frac{1}{z}-\frac{1}{x}=c.$

Univ. of State of N. Y.

2. Solve the equations,
$$2x+5y=85,$$
$$2y+5z=103,$$
$$2z+5x=57.$$
Vassar.

3. Solve the following set of equations:
$$x+y=-1.$$
$$x+3y+2z=-4.$$
$$x-y+4z=5.$$
Cornell.

4. Solve the equations:
$$\frac{7x+6}{11}+y-16=\frac{5x-13}{2}-\frac{8y-x}{5},$$
$$3(3x+4)=10y-15.$$
Mass. Inst. Tech.

5. Draw the lines represented by the equations
$$3x-2y=13 \text{ and } 2x+5y=-4,$$
and find by algebra the coördinates of the point where they intersect. *Univ. of Cal.*

6. Solve the simultaneous equations:
$$x+y=a+b, \quad \frac{y+b}{x+a}=\frac{a}{b}$$
and verify your results. *Harvard.*

SYSTEM OF EQUATIONS, ONE OR BOTH QUADRATIC

203. 1. Solve the system:
$$\left.\begin{aligned} x^2+y^2+xy&=54\tfrac{1}{9}, \\ x-y&=6\tfrac{1}{3}. \end{aligned}\right\}$$
Univ. of Pa.

2. Solve $\begin{cases} x-y=2, \\ x^2y-xy^2=30. \end{cases}$ *Univ. of State of N. Y.*

3. Solve for t and u: $\frac{1}{t^2}+\frac{1}{u^2}=74,\quad \frac{1}{t}-\frac{1}{u}=2.$ *Univ. of Pa.*

4. Solve the equations

$$x^2-xy+y^2=7,$$
$$2x-3y=0.$$

Columbia.

5. Plot the following two equations, and find from the graphs the approximate value of their common solutions:

$$x^2+y^2=25.$$
$$4x^2+9y^2=144.$$

Columbia.

6. Solve the following pair of equations for x and y:

$$x^2+y^2=4,$$
$$x=(1+\sqrt{2})y-2.$$

Cornell.

7. Solve $\left\{\begin{array}{l} x^2+3y^2=10, \\ x-y=2, \end{array}\right\}$ and sketch the graphs.

Mass. Inst. Tech.

BINOMIAL THEOREM

204. 1. Write in the simplest form the last three terms of the expansion of $(4a^{\frac{3}{2}}-a^{\frac{1}{2}}x^{\frac{1}{3}})^8$. *Yale.*

2. Expand $(1-\sqrt{2})^5$ by the Binomial Theorem.

Univ. of Pa.

3. Write out by the binomial theorem the first *five* terms of $\left(2a-\frac{b^2}{3}\right)^6$. *Univ. of State of N. Y.*

4. Find the seventh term of $\left(a+\frac{1}{a}\right)^{13}$. *Columbia.*

5. In the expansion of $\left(2x+\frac{1}{3x}\right)^6$ the ratio of the fourth term to the fifth is $2:1$. Find x. *Princeton.*

PROBLEMS

205. 1. Two cars of equal speed leave A and B, 20 mi. apart, at different times. Just as the cars pass each other an accident reduces the power and their speed is decreased 10 mi. per hour. One car makes the journey from A to B in 56 min., and the other from B to A in 72 min. What is their common speed? *Yale.*

2. The sum of two numbers is 13, and the sum of their cubes is 910. Find the smaller number, correct to the second decimal place. *Vassar.*

3. A man arranges to pay a debt of $3600 in 40 monthly payments which form an A. P. After paying 30 of them he still owes $\frac{1}{3}$ of his debt. What was his first payment? *Princeton.*

4. A page is to have a margin of 1 inch, and is to contain 35 square inches of printing. How large must the page be, if the length is to exceed the width by 2 inches? *Mount Holyoke.*

5. Insert 10 arithmetical means between 10 and 61, and find the sum of the entire series. *Univ. of Pa.*

6. A man bought a number of cattle which cost him in all $672. If each head had cost him $4 less, he would have been able to buy 3 more. How many did he buy and at what price? *Univ. of Pa.*

7. A boy is 5 years older than his sister and $\frac{1}{6}$ as old as his father; the sum of the ages of all three is 51. Find the age of the father. *Univ. of State of N. Y.*

8. Find two consecutive numbers whose product is 306. *Univ. of State of N. Y.*

9. A man engaged to work a days on these conditions: for each day he worked he was to receive b cents, and for each day he was idle he was to forfeit c cents. At the end of a days he received d cents. How many days was he idle? *Univ. of Pa.*

10. For what values of m will the roots of the equation $(m + \frac{5}{2})x^2 - 2(m + 1)x + 2 = 0$ be equal?

Form the equation whose roots are $\frac{2}{3}$ and $\frac{1}{2}$. *Univ. of Pa.*

11. If a number of two digits be divided by the product of its digits, the quotient will be 6. If 9 be added to the number, the sum will be equal to the number obtained by interchanging the digits. What is the number? *Univ. of Pa.*

12. A and B each shoot thirty arrows at a target. B makes twice as many hits as A, and A makes three times as many misses as B. Find the number of hits and misses of each.

Univ. of Cal.

13. The sides of a triangle are a, b, c. Calculate the radii of the three circles having the vertices as centers, each being tangent externally to the other two. *Harvard.*

14. The force P necessary to lift a weight W by means of a certain machine is given by the formula

$$P = a + bW,$$

where a and b are constants depending on the amount of friction in the machine. If a force of 7 pounds will raise a weight of 20 pounds, and a force of 13 pounds will raise a weight of 50 pounds, what force is necessary to raise a weight of 40 pounds?

(First determine the constants a and b.) *Harvard.*

15. Find the sum of 8 terms of the progression

$$5 + 3\tfrac{1}{3} + 2\tfrac{2}{9} + \cdots.$$ *Harvard.*

16. How many terms must be taken in the series 2, 5, 8, 11, $\cdots$, so that the sum shall be 345? *Mass. Inst. Tech.*

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